Algebraic Geometry

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1 Sheaves

In this section, we will focus on how to understand geometric spaces via the (nice) functions on them. This is made concise by something called **sheaf**.

1.1 Motivating example: The sheaf of differentiable functions

We consider differentiable functions on the topological space $X = \mathbb{R}^n$, i.e. the functions of the form $\mathbb{R}^n \to \mathbb{R}$ that is differentiable. There are two key facts that we need to first recognise:

- On each open set $U \subset X$, we have a ring of differentiable functions, denoted by $\mathcal{O}(U)$, where
  - addition is given by $(f + g)(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})$;
  - multiplication is given by $(fg)(\mathbf{x}) := f(\mathbf{x}) \cdot g(\mathbf{x})$.
- If we have a differentiable function on an open set, we can restrict it to a smaller open set, i.e. if $U \subset V$ then we have a "restriction map"

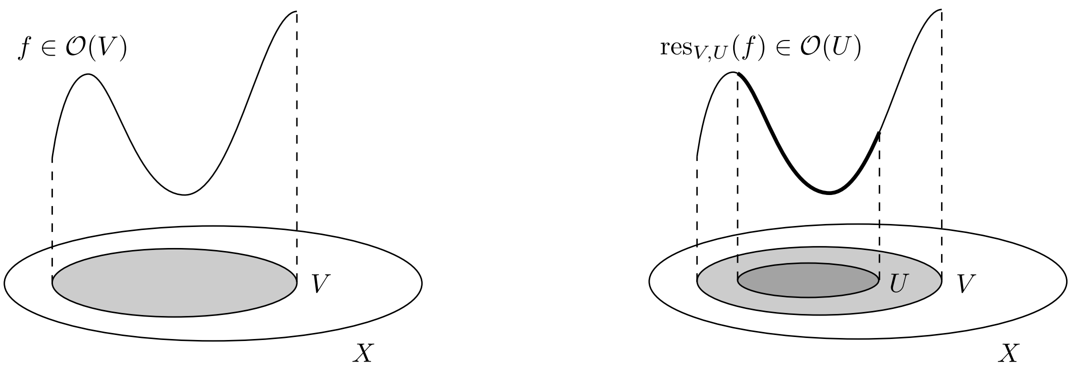
  $$\mathrm{res}_{V,U} : \mathcal{O}(V) \to \mathcal{O}(U).$$

  The restriction maps need to "naturally" commute, i.e. if $U \subset V \subset W$, then the following commutes:

  $$\begin{tikzcd} \mathcal{O}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{O}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{O}(U) & \end{tikzcd}$$

  This reads as "restricting from big to small is equal to restricting from big to medium, then to small".

To visualise the restriction maps, it is best to "plot" the graphs, so we obtain the following figures:



To this step, we have already defined what's called a **presheaf** (of differentiable functions, on the topological space $\mathbb{R}^n$), which is exactly $\mathcal{O}(U)$. Notice that this assigns each open set to a ring of functions.

Now to proceed to the actual sheaf we need two more technical observation:

- Let $f_1, f_2 \in \mathcal{O}(U)$, and let U_i be open sets such that $U = \cup_{i \in I} U_i$, i.e. $\{U_i\}$ covers U. Suppose that f_1 and f_2 agree on each of these U_i, then **they must have been the same function to begin with**.

  In other words, if $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$ for all i, then $f_1 = f_2$.
- Suppose we have the same U and open cover as above. Let $f_i \in \mathcal{O}(U_i)$ and assume they agree on the pairwise overlaps. Then **they can be "glued together" to form one differentiable function on U.**

  In other words, if $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$.

These two observations on differentiable functions are such fundamental to the nature of $\mathbb{R}^n$, so they eventually became the axioms in the definition of a **sheaf** as we will see later on. Intuitively, the first axiom says that there is at most one way to glue together differentiable functions, and the second says there is at least one way to glue.

The major takeaway from this motivation is as follows:

! Keypoint

The entire example would have worked with continuous functions, or smooth functions, or just plain functions.

So these classes of "nice" functions share common properties, and indeed these are all generalised to the properties of sheaves that we shall discuss later. But before we do anything, let's talk about another important concept first:

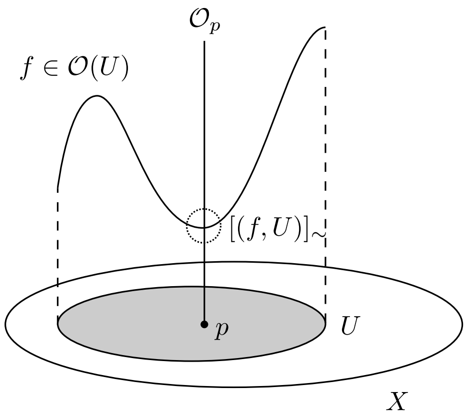
Definition 1.1.1 (Stalk and germ of a differentiable function)

Define the **stalk** $\mathcal{O}_p$ at a point p to be the set

$$\{(f, \text{open } U) : p \in U, f \in \mathcal{O}(U)\}$$

modulo the equivalence relation that $(f, U) \sim (g, V)$ if there is some open set $W \subset U \cap V$ containing p such that $f|_W = g|_W$. In other words, two functions that are the same in an open neighbourhood of p have the same germ. The equivalence classes are called **germs**.

The correct way to think about a germ is to think of it as a "shred" of some section near p, i.e. a germ in $\mathcal{O}_p$ stores the data of some small region near p, rather than just the value of $f(p)$:



Thus a germ stores a "small piece of f", and if another function shares the same piece then they have the same germ. The stalk then stores all possible small pieces around p.

Now the stalk is naturally a ring: we can add two germs via $[(f, U)]_\sim + [(g, V)]_\sim := [(f|_{U\cap V} + g|_{U\cap V}, U \cap V)]_\sim$ and similarly for multiplication. These operations are well-defined for if $\widehat{f} \sim f$ where they agree on some open neighbourhood W of p) and $\widehat{g} \sim g$ where they agree on some open neighbourhood W' of p, then $\widehat{f} + \widehat{g} \sim f + g$ on $U \cap V \cap W \cap W'$.

Also, notice that if $p \in U$ we have a map $\mathcal{O}(U) \to \mathcal{O}_p$ by $f \mapsto [(f, U)]_\sim$. This turns out to correspond to the fact that **germs are colimits** (of all $\mathcal{O}(U)$) in terms of categories. Another fact is as follows:

Lemma 1.1.2 ($\mathcal{O}_p$ is a local ring)

Let $\mathfrak{m}_p$ be the ideal containing germs vanishing at p. Then $\mathfrak{m}_p$ is the only maximal ideal of $\mathcal{O}_p$.

Proof. We shall show that all elements of $\mathcal{O}_p \setminus \mathfrak{m}_p$ are invertible. Let $[(f, U)]_\sim \in \mathcal{O}_p \setminus \mathfrak{m}_p$, with representative (f, U). Then $f(p) \neq 0$ by definition. By continuity, there is an open neighbourhood $V \subseteq U$ such that $f(x) \neq 0$ for all $x \in V$. Notice that $(f, U) \sim (f, V)$ since f and f clearly agree on $V \subseteq U \cap V = V$.

Now, $[(1/f, V)]_\sim \in \mathcal{O}_p$ as V is an open neighbourohod of p and $1/f$ is differentiable. So

$$[(f,U)]_\sim \cdot [(1/f,V)]_\sim = [(f,V)]_\sim \cdot [(1/f,V)]_\sim = [(1,V)]_\sim = 1,$$

which implies the desired result. $\square$

In this case, the value of a function (or a germ) at a point can be interpreted as an element of $\mathcal{O}_p/\mathfrak{m}_p \cong \mathbb{R}$. It turns out again that this only works for a special class of space, namely the **locally ringed spaces** which we will see later.

1.2 Sheaf and Presheaf

We shall now formally define what sheaves and presheaves are. Again, they are very much based on the motivation from above, but the general idea is to generalise "differentiable" functions on U to all kinds of functions, or even just a set. To be concrete, we will define sheaves of sets, but the category Sets can be replaced by any category, including abelian groups Ab, k-vector spaces Vec_k, and so on.

We again start from the definition of presheaves:

Definition 1.2.1 (Presheaf)
A **presheaf** $\mathcal{F}$ on a topological space X is the following data:

- For each open set $U \subseteq X$, we have a set $\mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**.
- For each inclusion $U \subseteq V$ of open sets, we have a **restriction map** $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$.

with two requirements:

- The map $\mathrm{res}_{U,U}$ is the identity $\mathrm{id}_{\mathcal{F}(U)}$.
- If $U \subseteq V \subseteq W$, then the restriction maps in the following diagram commute:

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{F}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{F}(U) & \end{tikzcd}$$

By convention, if "over U" is omitted, then it is implicitly taken to be X, i.e. **sections of $\mathcal{F}$** means sections of $\mathcal{F}$ over X, which are also called **global sections**. And if we choose $\mathcal{F}(U) = \mathcal{O}(U)$ (i.e. choosing the presheaf to be differentiable functions), we obtain the example in the previous section.

Remark. Categorically, a presheaf is exactly the data of a contravariant functor from the category of open sets of X to Sets as morphisms in the category of open sets are precisely inclusion maps, i.e. $\mathcal{F}$ is a contravariant functor

$$\mathcal{F} : \mathsf{OpenSets}(X)^{\mathrm{op}} \to \mathsf{Sets}.$$

Again, we define stalk and germs by extending the idea from above, but this time with two equivalent ways:

Definition 1.2.2 (Stalk and germs, constructive)
Define the **stalk** $\mathcal{F}_p$ at a point p to be the set

$$\{(f, \text{open } U) : p \in U, f \in \mathcal{F}(U)\}$$

modulo the equivalence relation that $(f,U) \sim (g,V)$ if there is some open set $W \subset U \cap V$ containing p such that $\mathrm{res}_{U,W}(f) = \mathrm{res}_{V,W}(g)$. The equivalence classes are called **germs**.

A useful equivalent definition of a stalk is by considering it categorically:

> **Definition 1.2.3 (Stalk and germs, categorical)**
>
> A **stalk** of $\mathcal{F}$ at a point p is a colimit of all $\mathcal{F}(U)$ over open neighbourhoods U of p:
>
> $$\mathcal{F}_p = \varinjlim \mathcal{F}(U).$$
>
> The image of $f \in \mathcal{F}(U)$ under the colimit map is then called the **germ** of f at p.

Remark. Expanding the categorical terms, the above definition means that $\mathcal{F}_p$ is an object satisfying the universal property: there is a map $\mathcal{F}(U) \to \mathcal{F}_p$ for every U such that the following diagram commutes (for all choices of U, V):

$$\begin{tikzcd}
\mathcal{F}_p & \\
\mathcal{F}(U) \arrow[u] \arrow[r, "\mathrm{res}_{U,V}"'] & \mathcal{F}(V) \arrow[ul]
\end{tikzcd}$$

and that if there are any other objects Z in the position of $\mathcal{F}_p$ making the diagram commute, then there is a unique map $\mathcal{F}_p \to Z$. By standard argument, this makes the stalk unique up to unique isomorphism if it exists.

The two definitions are equivalent since the index category (the category of open sets) is a **filtered** set: for any two open sets there is always a third open set contained in both. Thus the second definition actually allows us to define stalks for sheaves of sets, groups, rings, and other things for which colimits exist.

Finally, sheaves:

> **Definition 1.2.4 (Sheaf)**
>
> A **sheaf** is a presheaf that satisfies the following two axioms:
>
> Identity — If $\{U_i\}$ is an open cover of U, $f_1, f_2 \in \mathcal{F}(U)$ and $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$ for all i, then $f_1 = f_2$.
>
> Gluing — If $\{U_i\}$ is an open cover of U and $f_i \in \mathcal{F}(U_i)$ for all i such that $\mathrm{res}_{U_i,U_i \cap U_j} f_i = \mathrm{res}_{U_j,U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

Again, this means that we have a unique way to glue together sections which agree on their overlaps. We may interpret the two axioms categorically:

> **Proposition 1.2.5**
>
> Let $\mathcal{F}$ be a sheaf, $\{U_i\}$ be the set of all open neighbourhoods of p and U be the union of all U_i. Then
>
> $$\mathcal{F}(U) = \varprojlim \mathcal{F}(U_i).$$

Proof. The diagram is:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[d, "\mathrm{res}"'] \arrow[dr, "\mathrm{res}"] & \\
\mathcal{F}(U_i) \arrow[r, "\mathrm{res}"] & \mathcal{F}(U_j) \\
Z \arrow[u, "f_i"] \arrow[ur, "f_j"'] \arrow[uu, dashed, bend left, "\phi"] &
\end{tikzcd}$$

The upper triangle commutes since all maps are restriction maps. The gluing axiom guarantees that there is at least one map $\phi : Z \to \mathcal{F}(U)$ by $z \mapsto (\text{gluing } f_i(z))$ as $\{U_i\}$ is an open cover of U.

Now the identity axiom then guarantees the map is unique for if ϕ' is another such map, then for any i we have

$$\mathrm{res}_{U,U_i}(\phi'(z)) = f_i(z) = \mathrm{res}_{U,U_i}(\phi(z))$$

where the first equality is by the fact that ϕ' makes the diagram commute and the second is by definition of ϕ. Thus $\phi'(z) = \phi(z)$ for any $z \in Z$ and so $\mathcal{F}(U)$ is the desired limit. □

Example 1.2.6

Here are more examples for sheaves:

- Pre-sheaves of plain / continuous / differentiable / smooth real functions are all sheaves, since continuity is a *local* property (meaning that we only have to look at small open neighbourhoods at once).
- The pre-sheaf of constant real functions is **not** a sheaf in general, since we can pick two disjoint open sets U_1, U_2, so that the constant function 1 on U_1 and the constant function 2 on U_2 is not gluable.
- Similarly, the pre-sheaf of bounded real functions is also **not** a sheaf. Indeed, take $U_i = (i-2, i+2)$ and define $f_i \in \mathcal{F}(U_i)$ by $f_i(x) = x$, so $\mathbb{R} = \cup_{i\in\mathbb{Z}} U_i$. Now suppose there exists a bounded real function f such that $\mathrm{res}_{U,U_i} f = f_i$ for all i, then $\sup |f| < N$ for some $N \in \mathbb{R}$. Yet

  $$f(N) = \mathrm{res}_{U,U_N} f(N) = f_N(N) = N > \sup |f|$$

  which is absurd. Thus this example fails the gluing axiom as well.

The examples should ring a bell in your head about how to view a sheaf correctly:

! Keypoint

Sheaves are presheaves for which $\mathcal{F}$ is a **local** property.

But more exotic examples of sheaves exist. One of them is the following important example, called the **skyscraper sheaf**, where the sheaves are not classes of functions anymore:

Proposition 1.2.7 (Skyscraper sheaf)

Suppose X is a topological space with $p \in X$ and S is a set. Then

$$i_{p,*}S(U) = \begin{cases} S & \text{if } p \in U \\ \{e\} & \text{if } p \notin U \end{cases}$$

forms a sheaf. Here $\{e\}$ is any one-element set.

Proof. Being the first formal proof of a sheaf, let's actually verify all axioms. Its restriction maps can be given by

$$\mathrm{res}_{V,U} : i_{p,*}S(V) \to i_{p,*}S(U) = \begin{cases} \mathrm{id}_S & \text{if } p \in U \\ c_e & \text{if } p \notin U \end{cases}$$

where c_e denote the map which sends anything to e, which clearly works by simply expanding the $i_{p,*}S(V)$ and $i_{p,*}S(U)$ in both cases. Now it is a presheaf because

- $\mathrm{res}_{U,U} = \mathrm{id}_{i_{p,*}S(U)}$, which is trivial since in this case c_e reduces to $\mathrm{id}_{\{e\}}$ if $p \notin U$.
- If $U \subseteq V \subseteq W$, then

  $$\begin{tikzcd} i_{p,*}S(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & i_{p,*}S(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & i_{p,*}S(U) & \end{tikzcd}$$

  commutes, for if $p \in U$ then everything is S so all maps are id_S and if $p \notin U$ then both $\mathrm{res}_{W,U}$ and $\mathrm{res}_{V,U}$ would be c_e, so everything is mapped to e.

It remains to check the two axioms for it to be a sheaf. Let $\{U_i\}$ be an open cover of U.

Identity Suppose we have $f_1, f_2 \in i_{p,*}S(U)$ and $\operatorname{res}_{U,U_i} f_1 = \operatorname{res}_{U,U_i} f_2$ for all i.

- If $p \in U$ then $f_1, f_2 \in S$ and there exists a U_i containing p. So

$$f_1 = \operatorname{id}_S f_1 = \operatorname{res}_{U,U_i} f_1 = \operatorname{res}_{U,U_i} f_2 = \operatorname{id}_S f_2 = f_2.$$

- If $p \notin U$ then we must have $f_1 = f_2 = e$.

Thus in both cases we have $f_1 = f_2$.

Gluing Suppose we have $f_i \in i_{p,*}S(U_i)$ for all i such that $\operatorname{res}_{U_i,U_i\cap U_j} f_i = \operatorname{res}_{U_j,U_i\cap U_j} f_j$ for all i, j.

- If $p \in U$ then again there exists U_i containing p. Choose $f = f_i \in S = i_{p,*}S(U)$. This works since if U_j contains p we have

$$\operatorname{res}_{U,U_j} f = \operatorname{id}_S f = \operatorname{id}_S f_i = \operatorname{res}_{U_i,U_i\cap U_j} f_i = \operatorname{res}_{U_j,U_i\cap U_j} f_j = \operatorname{id}_S f_j = f_j$$

  while if U_j does not contain p then $f_j = e$ by definition and so

$$\operatorname{res}_{U,U_j} f = e = f_j.$$

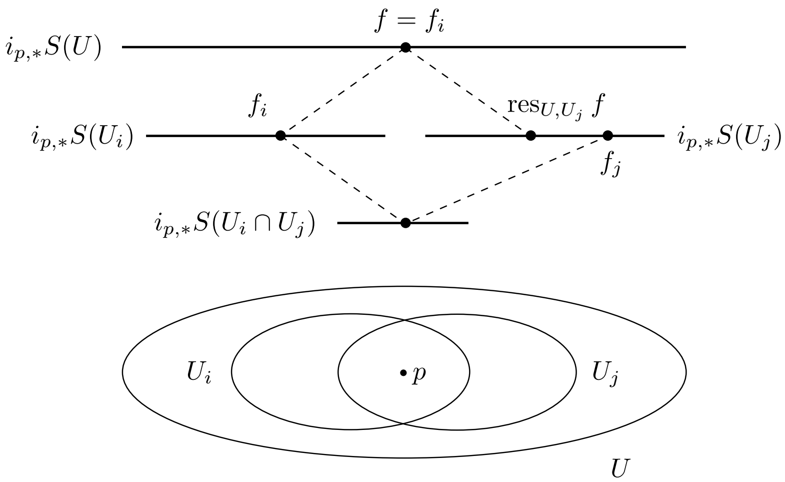
- If $p \notin U$ then $p \notin U_i$ for all i so $f = e$ works.

Again, in both cases we have constructed a glued up function.

Hence, $i_{p,*}S$ is a sheaf as desired. $\square$

The proof above is somewhat convoluted since this time we are not dealing with functions, instead just general sets as sheaves. But the idea is simple: we just have to check the axioms. In the future, we shall only mention the key steps to prove an object is a sheaf.

It might also be easier to understand the proof by drawing pictures, such as the figure below showcasing why the gluing argument for $p \in U$ works:



where the dashed lines are restriction maps but they are all in fact id_S, so connected dots are equal.

Remark. The notation $i_{p,*}S$ is admittedly hideous, but it would be explained in Example 1.2.14.

We mentioned that constant functions are in general not a sheaf but they form a presheaf. This idea is generalised:

Definition 1.2.8 (Constant presheaf)

Let X be a topological space and S be a set. Define $\underline{S}_{\text{pre}}(U) = S$ for all open sets U, then $\underline{S}_{\text{pre}}$ is a presheaf called the **constant presheaf** (associated to S).

Similarly, this is generally not a sheaf (for instance, by observing that the sections over the empty set must be the final object, i.e. a one-element set). However, we can do better:

Definition 1.2.9 (Constant sheaf)

Let X be a topological space and S be a set. Define $\underline{S}(U)$ to be the set of functions $U \to S$ which are **locally constant**, i.e. there is an open neighbourhood of p where the function is constant. Then $\underline{S}$ is a sheaf, called the **constant sheaf** (associated to S).

Caution: Constant sheaves consist of **locally** constant functions, not constant functions.

It is easy to see why $\underline{S}$ is a presheaf. To see why it is a sheaf, we can view S as a topological space with the discrete topology, then locally constant functions correspond to continuous functions $U \to S$.

Thus it suffices to show that continuous functions form a sheaf. We briefly mentioned this before, but this is important so we shall independently state it here:

Proposition 1.2.10 (Morphisms glue)

Let X, Y be topological spaces. Then continuous maps to Y form a sheaf of sets on X.

Proof. Pre-sheaf and the identity axiom are easy to verify. It remains show gluability. Let $\{U_i\}$ be an open cover of U and f_i be continuous maps from U_i to Y. We claim that $f(x) := f_i(x)$ where $x \in U_i$ works as a glued function. It is well-defined since if $x \in U_i, U_j$ then $f_i(x) = f_j(x)$ by assumption.

Now let $x \in U$ and $V \subseteq Y$ be an open neighbourhood of $f(x)$. As $x \in U$ we also have $x \in U_i$ for some i, so $f(x) = f_i(x)$. By continuity of f_i, there is some open neighbourhood $U_i' \subseteq U_i$ such that $f_i(U_i') \subseteq V$. But then

$$f(U_i') = f_i(U_i') \subseteq V,$$

i.e. f is also continuous. $\square$

A simple corollary is as follows:

Corollary 1.2.11 (Sheaf of sections of a map)

Let $\mu : Y \to X$ be a continuous map. Then the **sections of μ**, defined by

$$\mathcal{F}(U) = \{\text{continuous } s : U \to Y : \mu \circ s = \mathrm{id}_U\}$$

forms a sheaf.

Indeed, continuity is gluable, so we just have to check the newly added condition, which is easy to verify.

Furthermore, this particular example actually justifies why elements in $\mathcal{F}(U)$ are called sections: **every sheaf is the sheaf of sections** over a special space, called the **étalé space**.

Definition 1.2.12 (Étalé spaace of a (pre)sheaf)

Let $\mathcal{F}$ be a presheaf on a topological space X. Define F to be the disjoint union of all the stalks of $\mathcal{F}$, and $\mathcal{T}$ to be the topology with basis as subsets $\{(p, s_p) : p \in U\} \subseteq F$ over all $s \in \mathcal{F}(U)$ (here, s_p denotes the germ of s at p). The topological space $(F, \mathcal{T})$ is then called the **étalé space** of $\mathcal{F}$.

Notice that there is a natural map $\pi : F \to X$ by sending a germ s_p at p to p (which is also a local homeomorphism). It can then be checked that the sheaf of sections of π is precisely the presheaf $\mathcal{F}$ we start with.

We proceed by introducing an important concept:

Definition 1.2.13 (Pushforward sheaf)

Suppose $\pi : X \to Y$ is a continuous map and $\mathcal{F}$ is a presheaf on X. Then the **pushforward** of $\mathcal{F}$ by π,

$$\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$$

for open subsets V of Y defines a presheaf on Y, and is a sheaf if $\mathcal{F}$ is.

We omit the details of checking that it is a presheaf/sheaf here since it is quite simple. As promised before, we have the following example:

Example 1.2.14 (Skyscraper is the pushforward of constant by inclusion map)

Let $i_p : \{p\} \to X$ be the inclusion morphism from the one-point space p to a topological space X. Then the pushforward of the constant sheaf $\underline{S}$ is

$$(i_p)_*\underline{S}(U) = \underline{S}(i_p^{-1}(U)) = \begin{cases} \underline{S}(\{p\}) & \text{if } p \in U \\ \underline{S}(\phi) & \text{if } p \notin U \end{cases}$$

But this is equal to

$$(i_p)_*\underline{S}(U) = \begin{cases} S & \text{if } p \in U \\ \{e\} & \text{if } p \notin U \end{cases}$$

since $\{p\}$ is open (and so $\underline{S}(\{p\})$ must be constant), and that $\underline{S}(\phi)$ must be a final object, i.e. a one-element set.

This gives precisely the skyscraper sheaf as defined before, which explains the notation.

Moreover, we have the following:

Proposition 1.2.15 (Pushforward induces maps of stalks)

Suppose $\pi : X \to Y$ is a continuous map and $\mathcal{F}$ is a sheaf on X. If $\pi(p) = q$, then there is a natural morphism of stalks $(\pi_*\mathcal{F})_q \to \mathcal{F}_p$.

Proof. We provide two proofs of this based on the two definitions of stalks.

Constructive The morphism is given by $[(s, V)]_\sim \mapsto [(s, \pi^{-1}(V))]_\sim$. The image is a germ at p since $q \in V$ implies $p \in \pi^{-1}(V)$ and $s \in \pi_*\mathcal{F}(V)$ implies $s \in \mathcal{F}(\pi^{-1}(V))$.

Categorical We have

$$(\pi_*\mathcal{F})_q = \varinjlim_{q \in V} \pi_*\mathcal{F}(V) = \varinjlim_{q \in V} \mathcal{F}(\pi^{-1}(V)),$$

but notice that $q \in V$ implies $p \in \pi^{-1}(V)$, so $\mathcal{F}(\pi^{-1}(V))$ for $q \in V$ is a "subsystem" of $\mathcal{F}(U)$ for $p \in U$. Hence

$$\begin{tikzcd}
(\pi_*\mathcal{F})_q \arrow[dd, dashed, bend right] & \\
\mathcal{F}(\pi^{-1}(V_i)) \arrow[u] \arrow[r] \arrow[d] & \mathcal{F}(\pi^{-1}(V_j)) \arrow[ul] \arrow[dl] \\
\mathcal{F}_p &
\end{tikzcd}$$

commutes. By the universal property of $(\pi_*\mathcal{F})_q$, there is then a map $(\pi_*\mathcal{F})_q \to \mathcal{F}_p$. □

The final example of sheaves that we will cover is the structure sheaf of a ringed space.

Definition 1.2.16 (Ringed spaces)

Suppose $\mathcal{O}_X$ is a sheaf of rings on a topological space X (i.e. $\mathcal{O}_X(U)$ is a ring for all open U). Then $(X, \mathcal{O}_X)$ is called a **ringed space**, and the sheaf $\mathcal{O}_X$ is called the **structure sheaf** of the ringed space. Sections of $\mathcal{O}_X$ over an open subset U is further called the **functions on** U.

It is worth mentioning that usually when we use the symbol $\mathcal{O}_X$ we immediately refer to a structure sheaf of a ringed space X. The stalk of $\mathcal{O}_X$ at point p would be written as $\mathcal{O}_{X,p}$ (instead of $\mathcal{O}_{Xp}$).

We also naturally extend the concept of modules over a ring to modules over a structure sheaf:

Definition 1.2.17 ($\mathcal{O}_X$-modules)

An **$\mathcal{O}_X$-module** is a sheaf of abelian groups $\mathcal{F}$ with the following structure:

- $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module.
- If $U \subseteq V$, then the following diagram commute:

$$\begin{tikzcd}
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] \arrow[d, "\operatorname{res}^{\mathcal{O}_X}_{V,U} \times \operatorname{res}^{\mathcal{F}}_{V,U}"'] & \mathcal{F}(V) \arrow[d, "\operatorname{res}^{\mathcal{F}}_{V,U}"] \\
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"'] & \mathcal{F}(U)
\end{tikzcd}$$

where $\operatorname{res}^{\mathcal{O}_X}_{V,U}$ and $\operatorname{res}^{\mathcal{F}}_{V,U}$ represents the restriction maps of $\mathcal{O}_X$ and $\mathcal{F}$ respectively. (Essentially this is saying that restriction maps of $\mathcal{F}$ are compatible with that of $\mathcal{O}_X$.)

Caution: Although named as it is, a $\mathcal{O}_X$-module is not actually a module as $\mathcal{O}_X$ is not a ring

Recall that the notion of A-module generalises the notion of abelian groups as it is the same thing as a $\mathbb{Z}$-module. In this case, the notion of $\mathcal{O}_X$-module generalises the **sheaf of abelian groups** as the latter is the same thing as a $\underline{\mathbb{Z}}$-module where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$.

Finally, a $\mathcal{O}_X$-module induces properties on its stalks:

Proposition 1.2.18

If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then for each $p \in X$, $\mathcal{F}_p$ is an $\mathcal{O}_{X,p}$-module.

Proof. Notice that $(\mathcal{F}_p, +)$ is automatically an abelian group as $\mathcal{F}_p = \varinjlim \mathcal{F}(U)$ and $\mathcal{F}(U) \in \mathsf{Ab}$. Now notice that

$$\mathcal{O}_{X,p} \times \mathcal{F}_p = \varinjlim \mathcal{O}_X(U) \times \mathcal{F}(U),$$

so the diagram

$$\begin{tikzcd}
\mathcal{O}_{X,p} \times \mathcal{F}_p \arrow[ddd, dashed, bend right] & \\
\mathcal{O}_X(U_i) \times \mathcal{F}(U_i) \arrow[u] \arrow[r] \arrow[d] & \mathcal{O}_X(U_j) \times \mathcal{F}(U_j) \arrow[ul] \arrow[d] \\
\mathcal{F}(U_i) \arrow[r] \arrow[d] & \mathcal{F}(U_j) \arrow[dl] \\
\mathcal{F}_p &
\end{tikzcd}$$

commutes. Hence there is a unique map (scalar product) from $\mathcal{O}_{X,p} \times \mathcal{F}_p$ to $\mathcal{F}_p$ by the universal property. $\square$

This concludes our examples of sheaves for now.

1.3 Morphism of presheaves and sheaves

Whenever one defines a new mathematical object, it is natural to consider morphisms between them. This section would precisely be dedicated to that.

Definition 1.3.1 (Morphism of presheaves and sheaves)

A **morphism of presheaves** of sets $\phi : \mathcal{F} \to \mathcal{G}$ on X is the data of maps $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all U such that restriction is preserved, i.e. the diagram

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\mathrm{res}_{V,U}"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "\phi(U)"'] & \mathcal{G}(U) \end{tikzcd}$$

commutes.

A **morphism of sheaves** is defined identically where the morphism is just that of the underlying presheaves.

This can be understood categorically: A morphism of presheaves on X is a **natural transformation** of functors. With the morphism defined, we would start to denote $\mathsf{Sets}_X, \mathsf{Ab}_X$, etc. as the category of sheaves of sets, abelian groups, etc. on a topological space X, and $\mathsf{Sets}_X^{\mathrm{pre}}$ etc. would denote the category of presheaves of sets, etc. on X.

Caution: Be careful: a presheaf is a functor, but presheaves also form a category. It is best to forget that presheaves are functors from now.

Example 1.3.2

A simple example of morphism of sheaves is the map from the sheaf of differentiable functions on $\mathbb{R}$ to the sheaf of continuous functions. The morphism $\phi(U)$ is just given by $f \mapsto f$ since the function is not changed, but this is a **forgetful** map since we are forgetting the functions are differentiable.

The following is a very important note:

Proposition 1.3.3

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X and $p \in X$. Then there is an induced morphism of stalks $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$, defined by $[(f, U)] \mapsto [(\phi(U)(f), U)]$.

Proof. We simply check that the map is well-defined. Indeed, suppose $(f, U) \sim (g, V)$ where they agree on $W \subseteq U \cup V$. Then the following diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(W) \arrow[r, "\phi(W)"] & \mathcal{G}(W) \\ \mathcal{F}(V) \arrow[r, "\phi(V)"'] \arrow[u] & \mathcal{G}(V) \arrow[u] \end{tikzcd}$$

commutes, so

$$\phi(U)(f)|_W = \phi(W)(f|_W) = \phi(W)(g|_W) = \phi(V)(g)|_W,$$

thus $(\phi(U)(f), U) \sim (\phi(V)(g), V)$ since they agree on W as well. $\square$