Algebraic Geometry

Part I: Varieties

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Disclaimer

This is a set of handouts aimed to cover the basics of classical algebraic geometry, or mainly the study of varieties. The content is heavily based on the amazing handout from Andreas Gathmann[1], Smith et al.'s "*An Invitation to Algebraic Geometry*"[2], and Evan Chen's "*An Infinitely Large Napkin*"[3], so all credits go to them.

Also typed for my first-year UROP, supervised by Prof. Johannes Nicaise.

[2] `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2021/alggeom-2021.pdf`

[3] `https://www.ufjf.br/fred_feitosa/files/2009/08/Universitext-Karen-E.-Smith-Pekka-Kek%C3%A4l%C3%A4inen-Lauri-Kahanp%C3%A4%C3%A4-William-Traves-An-invitation-to-algebraic-geometry-Springer-2000.pdf`

[3] `https://venhance.github.io/napkin/Napkin.pdf`

Contents

1 Affine Varieties

This section is devoted to introduce the first goal of classical algebraic geometry: studying solutions of polynomial equations over a field K (from now, K will always denote a fixed algebraically-closed field unless otherwise stated).

1.1 Definitions and Examples

In its easiest form, we want to consider the following type of objects:

Definition 1.1 (Affine variety)
We call

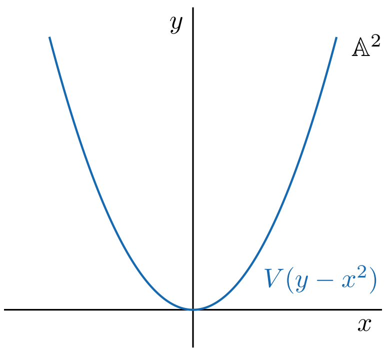
$$\mathbb{A}^n := \mathbb{A}^n_K := \{(a_1, \ldots, a_n) : a_i \in K \text{ for } i = 1, \ldots, n\}$$

the **affine n-space** over K. For a subset $S \subseteq K[x_1, x_2, \ldots, x_n]$ of polynomials, we call

$$V(S) := \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\} \subseteq \mathbb{A}^n$$

the **zero locus** of S. Subsets of $\mathbb{A}^n$ of this form are called **affine varieties**.

In other words, $V(S)$ is the set of points vanishing on *all* the polynomials in S. For example, a parabola is the zero locus of the polynomial $V(y - x^2)$:



Caution: As above, when drawing an affine variety V we will of course draw only its real points $V \cap \mathbb{R}^n$.

Note that $\mathbb{A}^n_K$ is just K^n as a set. We use two different notations here since K^n is also a K-vector space and a ring. We will usually use the notation $\mathbb{A}^n_K$ if we want to ignore these additional structures: for example, addition or scalar multiplication is not defined on $\mathbb{A}^n_K$.

Example 1.2 (Examples of affine varieties)

- The affine n-space itself is an affine variety, since $\mathbb{A}^n = V(0)$.
  Similarly, the empty set $\emptyset = V(1)$ is an affine variety.
- Any point $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is an affine variety, since $\{a\} = V(x_1 - a_1, \ldots, x_n - a_n)$.
- The two axes in $\mathbb{A}^2$ can be thought of as $V(xy)$; this is the set of points such that $x = 0$ or $y = 0$.

Let's start with some simple properties of the operator V:

Lemma 1.3

(a) For any $S_1 \subseteq S_2 \subseteq K[x_1, \ldots, x_n]$ we have $V(S_1) \supseteq V(S_2)$.

(b) For any $S_1, S_2 \subseteq K[x_1, \ldots, x_n]$ we have $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where $S_1 S_2 := \{fg : f \in S_1, g \in S_2\}$.

(c) If J is any index set and $S_i \subseteq K[x_1, \ldots, x_n]$ for all $i \in J$ then $\bigcap_{i \in J} V(S_i) = V\big(\bigcup_{i \in J} S_i\big)$.

In particular, finite unions and arbitrary intersections of affine varieties are again affine varieties.

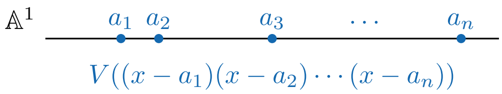
Proof. (a) If $x \in V(S_2)$, i.e. $f(x) = 0$ for all $f \in S_2$, then in particular $f(x) = 0$ for all $f \in S_1$, and hence $x \in V(S_1)$.

(b) ($\subseteq$). If $x \in V(S_1) \cup V(S_2)$ then $f(x) = 0$ for all $f \in S_1$ or $g(x) = 0$ for all $g \in S_2$. In both case this means $(fg)(x) = 0$ for all $f \in S_1$ and $g \in S_2$, i.e. $x \in V(S_1 S_2)$.

($\supseteq$). If $x \notin V(S_1) \cup V(S_2)$ then $x \notin V(S_1)$ and $x \notin V(S_2)$, i.e. there are $f \in S_1$ and $g \in S_2$ such that $f(x) \neq 0$ and $g(x) \neq 0$. This gives $(fg)(x) \neq 0$, so $x \notin V(S_1 S_2)$.

(c) We have $x \in \bigcap_{i \in J} V(S_i)$ if and only if $f(x) = 0$ for all $f \in S_i$ for all $i \in J$, which is the case if and only if $x \in V\left(\bigcup_{i \in J} S_i\right)$. $\square$

Thus, for example, any finite set of points is an affine variety. As an important realisation, in the special case of $\mathbb{A}^1$, the zero locus of any non-zero polynomial in $K[x]$ is already finite. Hence, **the affine varieties in $\mathbb{A}^1$ are exactly $\mathbb{A}^1$ itself and all finite sets**. So a typical affine variety in $\mathbb{A}^1$ looks like:



1.2 Ideals and Hilbert's Nullstellensatz

As you might have already noticed, a variety can be named by $V(\cdot)$ in multiple ways:

Motivation

For example, in $\mathbb{A}^2$,
$$\{(3,4)\} = V(x-3, y-4) = V(x-3, y-x-1).$$
That's a bit annoying, because in an ideal world we would have *one* name for every variety. To achieve this, a partial solution is to use ideals rather than sets. Than is, we consider
$$I = \langle x-3, y-4 \rangle = \{p \cdot (x-3) + q \cdot (y-4) : p, q \in K[x,y]\} \trianglelefteq K[x,y]$$
and look at $V(I)$. Note that in this case $\langle x-3, y-4 \rangle = \langle x-3, y-x-1 \rangle$ as ideals.

To make this precise, if f and g are polynomials that vanish on a certain subset $S \subseteq \mathbb{A}^n$, then $f+g$ and pf for any polynomial p also clearly vanishes on S. Thus for an affine variety $V(S)$ we can add $f+g$ and pf to S without changing its zero locus, so we always have
$$V(\langle S \rangle) = V(S),$$
where $\langle S \rangle$ is the ideal generated by S.

Thus, we can (and will) **only consider $V(I)$ where I is an ideal from now on**. Moreover, by Hilbert's Basis Theorem, any ideal in $K[x_1, \ldots, x_n]$ is finitely generated. This means that:

! Keypoint

Any affine variety can be written as the zero locus of finitely many polynomials.

You might ask whether the identification of a variety is unique now: that is, if $V(I) = V(J)$, does it follow that $I = J$? The answer is unfortunately no: even in $\mathbb{A}^1$, we have the counterexample
$$V(x) = V(x^2),$$
or in other words, the set of solutions to $x = 0$ is the same as that to $x^2 = 0$.

To fix this, we need an operation which "takes the ideal (x^2) and makes it into (x)"; this is exactly the following notion from commutative algebra:

Definition 1.4 (Radical of an ideal)

Let R be a ring. The **radical** of an ideal $I \subseteq R$, denoted $\sqrt{I}$, is defined by

$$\sqrt{I} := \{r \in R : r^k \in I \text{ for some integer } k \in \mathbb{N}\}.$$

If $I = \sqrt{I}$, we say the ideal I itself is **radical**.

Reformulating the results of Lemma 1.3 in terms of standard ideal-theoretic operations gives the following.

Lemma 1.5 (Properties of $V(\cdot)$)

For any ideals J, J_1, J_2 in $K[x_1, \dots, x_n]$ we have

(a) $V(\sqrt{J}) = V(J)$;

(b) $V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$;

(c) $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

Proof. (b) and (c) are just reformulations of Lemma 1.3, keeping in mind that $\sqrt{J_1 J_2} = \sqrt{J_1 \cap J_2}$ (for (b)); and $J_1 + J_2$ is the ideal generated by $J_1 \cup J_2$ (for (c)).

For (a), $(\subseteq)$ follows directly from Lemma 1.3(a) since $\sqrt{J} \supseteq J$. For the other inclusion, assume $x \in V(J)$ and $f \in \sqrt{J}$. Then $f^k \in J$ for some $k \in \mathbb{N}$, so that $f^k(x) = 0$, and hence also $f(x) = 0$. This means that $x \in V(\sqrt{J})$. $\square$

The motivation above is important since it reflects the basis of algebraic geometry: it relates geometric objects (varieties) to algebraic objects (ideals). In fact, the main goal of this chapter is to study this correspondence in detail. We can now introduce an operation that does the opposite job to V:

Definition 1.6 (Ideal of a subset of $\mathbb{A}^n$)

Let $X \subseteq \mathbb{A}^n$ be any subset. Then

$$I(X) := \{f \in K[x_1, \dots, x_n] : f(x) = 0 \text{ for all } x \in X\}$$

is called the **ideal of X** (note that this is indeed an ideal by the discussion above).

Remark. In analogy to Lemma 1.3(a), the ideal of a subset reverses inclusions: if $X_1 \subseteq X_2$ then $I(X_1) \supseteq I(X_2)$.

Note that $I(X)$ is always radical: if $f^k \in I(X)$ for some $f \in K[x_1, \dots, x_n]$ and $k \in \mathbb{N}$ then $f^k(x) = 0$ for all $x \in X$, so $f(x) = 0$ for all $x \in X$ as well and thus $f \in I(X)$. Hence, we have a correspondence:

$$\{\text{affine varieties}\} \underset{V(\cdot)}{\overset{I(\cdot)}{\rightleftarrows}} \{\text{radical ideals}\}$$

It is a central result of commutative algebra that this is actually a bijection:

Theorem 1.7 (Hilbert's Nullstellensatz)

(a) For any affine variety $X \subseteq \mathbb{A}^n$ we have $V(I(X)) = X$.

(b) For any ideal $J \trianglelefteq K[x_1, \dots, x_n]$ we have $I(V(J)) = \sqrt{J}$.

In particular, there is an *inclusion-reversing* bijection between the set of affine varieties of $\mathbb{A}^n$ and the set of radical ideals in $K[x_1, \dots, x_n]$.

Proof. Three of the four inclusions of (a) and (b) are actually easy:

(a) ($\supseteq$). If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$, and hence $x \in V(I(X))$.

(b) ($\supseteq$). If $f \in \sqrt{J}$ then $f^k \in J$ for some $k \in \mathbb{N}$. It follows that $f^k(x) = 0$ for all $x \in V(J)$, hence also $f(x) = 0$ for all $x \in V(J)$, and so $f \in I(V(J))$.

(a) ($\subseteq$). As X is an affine variety, $X = V(J)$ for some ideal J. Then $I(V(J)) \supseteq \sqrt{J} \supseteq J$ by (b) ($\subseteq$), so taking the zero locus we obtain $V(I(V(J))) \subseteq V(J)$ by Lemma 1.3(a). This means exactly that $V(I(X)) \subseteq X$.

Only the inclusion $I(V(J)) \subseteq \sqrt{J}$ of (b) is hard; a proof of this result from commutative algebra uses the convention that K is algebraically closed, and can be found in any book on commutative algebra. It is omitted here.

The additional bijection statement follows with the observation that $\sqrt{J} = J$ if J is radical, and that both operations reverse inclusions by Lemma 1.3(a) and the remark after Definition 1.6. □

We dedicate the rest of the section to showcase the power of the result:

Example 1.8 (Nullstellensatz on $\mathbb{A}^1$)

Continuing the $\mathbb{A}^1$ example from above:

- Let $J \trianglelefteq K[x]$ be a non-zero ideal. As $K[x]$ is a principal ideal domain, we have $J = \langle f \rangle$ for some $f = (x - a_1)^{k_1} \cdots (x - a_n)^{k_n}$ where $a_1, \dots, a_n \in \mathbb{A}^1$ are distinct points. The zero locus

$$V(J) = V(f) = \{a_1, \dots, a_n\} \subseteq \mathbb{A}^1$$

  then contains the data of the zeros of f, but **no longer the multiplicities** $k_1, \dots, k_n$. Consequently,

$$I(V(J)) = \sqrt{J} = \langle (x - a_1) \cdots (x - a_n) \rangle$$

  is just the ideal of all polynomials vanishing at $a_1, \dots, a_n$.

- If we had not assumed K to be algebraically closed, the Nullstellensatz does not hold even in simple examples: the prime (and hence radical) ideal $J = \langle x^2 + 1 \rangle \trianglelefteq \mathbb{R}[x]$ has empty zero locus in $\mathbb{A}^1_{\mathbb{R}}$, so

$$I(V(J)) = I(\emptyset) = \mathbb{R}[x] \neq J = \sqrt{J}.$$

Example 1.9 (Maximal ideals)

The ideal $J = \langle x_1 - a_1, \dots, x_n - a_n \rangle \trianglelefteq K[x_1, \dots, x_n]$ is maximal since $K[x_1, \dots, x_n]/J \cong K$ is a field, and hence also radical. As its zero locus is $V(J) = \{a\}$ for $a = (a_1, \dots, a_n)$, we conclude by Nullstellensatz that

$$I(\{a\}) = I(V(J)) = J = \langle x_1 - a_1, \dots, x_n - a_n \rangle.$$

In fact, points of $\mathbb{A}^n$ are clearly the **minimal non-empty varieties** in $\mathbb{A}^n$, so by the inclusion-reversing property they correspond exactly to the **maximal (proper) ideals** in $K[x_1, \dots, x_n]$. Hence

$$\{\text{points in } \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals in } K[x_1, \dots, x_n]\}$$

is a bijection again.

In other word, using Nullstellensatz we are able to deduce:

! Keypoint

Every maximal ideal of $K[x_1, \dots, x_n]$ is of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ (for K algebraically closed).

Again if K is not algebraically closed then this is not true, for example $J = \langle x^2 + 1 \rangle$ is maximal in $\mathbb{R}[x]$.

We might also translate properties of $V(\cdot)$ in Lemma 1.5 to corresponding properties of $I(\cdot)$:

Lemma 1.10 (Properties of $I(\cdot)$)
For any affine varieties X_1 and X_2 in $\mathbb{A}^n$ we have

(a) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$;

(b) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof. (a) A polynomial $f \in K[x_1, \ldots, x_n]$ is contained in $I(X_1 \cup X_2)$ if and only if $f(x) = 0$ for all $x \in X_1$ and all $x \in X_2$, which is the case if and only if $f \in I(X_1) \cap I(X_2)$.

(b) We have
$$I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)},$$
where the second equality comes from Lemma 1.5(c) and the others come from Nullstellensatz. □

As a consequence of the Nullstellensatz, we can also deduce a generalisation of the fact that any non-constant univariate polynomial has a zero over an algebraically closed field:

Proposition 1.11 (Weak Nullstellensatz)
Let $J \trianglelefteq K[x_1, \ldots, x_n]$ be an ideal. If $J \neq K[x_1, \ldots, x_n]$ then J has a zero, i.e. $V(J) \neq \emptyset$.

Proof. If not then $\sqrt{J} = I(V(J)) = I(\emptyset) = K[x_1, \ldots, x_n]$ by Nullstellensatz, so $1 \in \sqrt{J}$ and $1 \in J$, contradiction. □

Nullstellensatz also provides a trick to calculate radicals of polynomial ideals:

Example 1.12
Consider the ideal $J = \langle x^3 - y^6, xy - y^3 \rangle \trianglelefteq \mathbb{C}[x, y]$. To find its radical, we note that
$$x^3 - y^6 = (x - y^2)(x^2 + xy^2 + y^4) \quad \text{and} \quad xy - y^3 = y(x - y^2).$$
Hence $(x, y) \in V(J)$ is equivalent to either $x = y^2$ or $(x, y) = (0, 0)$. In any case $x = y^2$, so we conclude that
$$V(J) = \{(x, y) \in \mathbb{C}^2 : x = y^2\}$$
which clearly has ideal $\langle x - y^2 \rangle$. In other words, $\sqrt{J} = I(V(J)) = \langle x - y^2 \rangle$.

Another easy consequence is that polynomials and polynomial functions on $\mathbb{A}^n$ agree. This motivates the following discussion on coordinate rings.

Motivation
If $f, g \in K[x_1, \ldots, x_n]$ are two polynomials defining the same function on $\mathbb{A}^n$, i.e. $f(x) = g(x)$ for all $x \in \mathbb{A}^n$ then
$$f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{\langle 0 \rangle} = \langle 0 \rangle$$
and hence $f = g$ as polynomials. So $K[x_1, \ldots, x_n]$ can be thought of as the ring of polynomial functions on $\mathbb{A}^n$.

It is easy to generalise this idea to an affine variety $X \subseteq \mathbb{A}^n$: Two polynomials $f, g \in K[x_1, \ldots, x_n]$ define the same polynomial function on X if and only if $f - g \in I(X)$. Or in other words, the natural surjection
$$K[x_1, \ldots, x_n] \to \text{ring of polynomial functions on } X$$
has kernel $I(X)$, so the quotient ring $K[x_1, \ldots, x_n]/I(X)$ is isomorphic to the ring of polynomial functions on X.

Let's make this into a precise definition:

Definition 1.13 (Polynomial functions and coordinate rings)

Let $X \subseteq \mathbb{A}^n$ be an affine variety. A **polynomial function** on X is a map $X \to K$ of the form $x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$.

The ring of all polynomial functions on X is called the **coordinate ring** $A(X)$ of the affine variety X. By above, it is isomorphic to the quotient ring

$$A(X) \cong K[x_1, \dots, x_n]/I(X).$$

Remark. Note that the coordinate ring is not just a ring, but also a K-*algebra* (i.e. also an K-vector space such that its ring multiplication is K-bilinear). In fact, in the following we will often consider $A(X)$ as a K-algebra, despite its common name "coordinate ring".

Using coordinate rings, we can define the concepts introduced so far in a relative setting, i.e. consider zero loci of ideals in $A(Y)$ and varieties contained in Y for a fixed ambient affine variety Y that is not necessarily $\mathbb{A}^n$.

Definition 1.14 (Relative version of $V(\cdot)$ and $I(\cdot)$)

Let $Y \subseteq \mathbb{A}^n$ be a fixed affine variety.

(a) For a subset $S \subseteq A(Y)$ of polynomial functions on Y we define the **zero locus** as

$$V(S) := V_Y(S) := \{x \in Y : f(x) = 0 \text{ for all } f \in S\} \subseteq Y.$$

The subsets that are of this form are precisely the affine varieties **contained in** Y, so they are called the **affine subvarieties** of Y.

(b) For a subset $X \subseteq Y$ the **ideal** of X in Y is defined to be

$$I(Y) := I_Y(X) := \{f \in A(Y) : f(x) = 0 \text{ for all } x \in X\} \trianglelefteq A(Y).$$

Note that this is completely analogous to the definitions of $V(\cdot)$ and $I(\cdot)$ in the ambient space $\mathbb{A}^n$. With essentially the same arguments as before, all results in this chapter could be considered in the relative setting:

- In the same way as in Definition 1.13, we see that $A(X) \cong A(Y)/I_Y(X)$ for any affine subvariety X of Y.
- (**Relative Nullstellensatz**) As in Proposition 1.7, we have $V_Y(I_Y(X)) = X$ for any affine subvariety X of Y and $I_Y(V_Y(J)) = \sqrt{J}$ for any ideal $J \trianglelefteq A(Y)$, giving rise to a bijection

$$\{\text{affine subvarieties of } Y\} \longleftrightarrow \{\text{radical ideals in } A(Y)\}.$$

- With the same proofs, the properties of $V(\cdot)$ of Lemma 1.5 and the properties of $I(\cdot)$ of Lemma 1.10 hold in the relative setting as well.

1.3 The Zariski Topology

We proceed by endowing a topological structure on every variety V. Since our affine varieties all live in $\mathbb{A}^n$, all we have to do is put a suitable topology on $\mathbb{A}^n$, and then just view V as a subspace. However, rather than putting the standard Euclidean topology on $\mathbb{A}^n$, we put a much more bizarre topology.

Definition 1.15 (Zariski topology)

We define the **Zariski topology** on $\mathbb{A}^n$ to be the topology whose *closed* sets are those of the form

$$V(I) \quad \text{where } I \subseteq K[x_1, \dots, x_n].$$

Note that this is indeed a topology by Lemma 1.3.

The Zariski topology on any affine variety $X \subseteq \mathbb{A}^n$ is then the subspace topology on X, i.e. the closed sets are of the form $X \cap Y$ where Y is closed in $\mathbb{A}^n$. This agrees with the topology where the closed sets are affine subvarieties of X, i.e. the topology whose closed sets are of the form

$$V(I) \quad \text{where } I \subseteq A(X),$$

since the affine varieties of X are precisely the affine varieties contained in X, and the intersection of two affine varieties is again an affine variety.

Example 1.16 (Zariski topology on $\mathbb{A}^n$)

Let us determine the open sets of $\mathbb{A}^1$ and $\mathbb{A}^2$:

- The affine varieties of $\mathbb{A}^1$ are either $\mathbb{A}^1$ itself or a finite set (including $\emptyset$). Thus, the *open* sets of $\mathbb{A}^1$ are $\emptyset$ and $\mathbb{A}^1$ minus a finite collection (possibly empty) of points.

  Thus, a picture of a "typical" open set in $\mathbb{A}^1$ might be

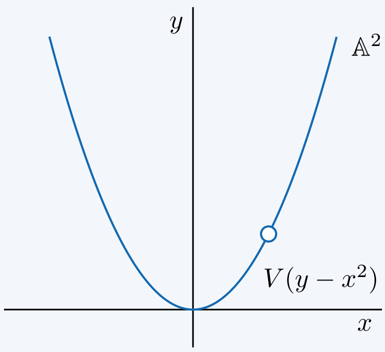
  i.e. it is everything except a few marked points.

- Similarly, in $\mathbb{A}^2$, the non-trivial closed sets are going to consist of finite unions of
  - closed curves, like $V(y - x^2)$, and
  - single points, like $V(x-3, y-4)$.

  Of course, the entire space $\mathbb{A}^2$ and the empty set $\emptyset$ are closed sets. So the open sets of $\mathbb{A}^2$ are the entire plane minus a finite collection of points and one-dimensional curves.

Example 1.17 (Zariski topology on an affine variety)

Let $X = V(y - x^2) \subseteq \mathbb{A}^2$ be a parabola, and let $U = V \setminus \{(1,1)\}$. We claim U is open in V:



Indeed, $\tilde{U} = \mathbb{A}^2 \setminus \{(1,1)\}$ is open in $\mathbb{A}^2$ (since it is the complement of the closed set $V(x-1, y-1)$), so $U = \tilde{U} \cap V$ is open in V. Note that on the other hand the set U is *not* open in $\mathbb{A}^2$.

Compared to the classical metric topology, the Zariski topology is certainly unusual; intuitively, we can say that:

! Keypoint

The non-empty Zariski open sets are *huge*.

This is an important difference between the two topologies. To be more precise,

- In the standard Euclidean topology, when we say "in an open neighbourhood of point p", one think of this as saying "in a small region around p".

- In the Zariski topology, saying something is true "in an open neighbourhood of point p" should be thought of as saying it is true "for virtually all points, other than those on certain curves".

Nonetheless, it will still be helpful to draw open neighbourhoods as circles in pictures that follow.

Example 1.18 (Some final bizarreness of the Zariski topology)

- (Continuity) If $f : \mathbb{A}^1 \to \mathbb{A}^1$ is any injective map, then f is automatically continuous since pre-images of finite subsets of $\mathbb{A}^1$ under f are again finite.

  But this statement is essentially useless, since we will not define morphisms of affine varieties as just being continuous maps in the Zariski topology. In fact, this example gives us a hint that we should not do so.

- (Product topology) The Zariski topology of an affine product variety $X \times Y$ is not the product topology: for example, the subset $V(x-y) = \{(a,a) : a \in K\} \subseteq \mathbb{A}^2$ is closed in the Zariski topology, but not in the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$ as it is not of the form

$$\text{finite set} \times \text{finite set}, \quad \text{finite set} \times \mathbb{A}^1, \quad \text{or } \mathbb{A}^1 \times \mathbb{A}^1.$$

1.4 Irreducibility and connectedness

Let us now start with the discussion of topological concepts that are actually useful in the Zariski topology. The first one concern components of an affine variety.

Definition 1.19 (Irreducible and connected spaces)

Let X be a topological space. We say

(a) X is **reducible** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subset X$. Otherwise X is called **irreducible**;

(b) X is **disconnected** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$. Otherwise X is called **connected**.

Caution: Note the strict inclusions of X_1, X_2 in both definitions.

Hence, for instance, the irreducible varieties of $\mathbb{A}^1$ are $\emptyset$, single points $V(x-a)$, and the entire line $\mathbb{A}^1$. As another example, the union of two axes $V(xy) \subset \mathbb{A}^2$ is reducible since it is the union of, well, the two axes $V(x)$ and $V(y)$, but it is connected since it is impossible to write it as a union of two **disjoint** closed subsets of $\mathbb{A}^2$.

We can also characterise these two notions algebraically in the Zariski topology:

Proposition 1.20

Let X be a disconnected affine variety, with $X = X_1 \cup X_2$ for two disjoint closed subsets $X_1, X_2 \subset X$. Then

$$A(X) \cong A(X_1) \times A(X_2).$$

Proof. By Lemma 1.10, as $X_1 \cup X_2 = X$ we obtain in $A(X)$

$$I(X_1) \cap I(X_2) = I(X_1 \cup X_2) = I(X) = \{0\}.$$

On the other hand, from $X_1 \cap X_2 = \emptyset$ we have in $A(X)$

$$\sqrt{I(X_1) + I(X_2)} = I(X_1 \cap X_2) = I(\emptyset) = \langle 1 \rangle,$$

and thus also $I(X_1) + I(X_2) = \langle 1 \rangle$. So by the Chinese Remainder Theorem, we conclude that

$$A(X) \cong A(X)/I(X_1) \times A(X)/I(X_2),$$

which by the relative definition of coordinate rings is exactly the statement of the proposition. $\square$

The following appears much more frequently than the result above:

Proposition 1.21

A non-empty affine variety X is irreducible if and only if $A(X)$ is an integral domain.

Proof. As X is non-empty, its coordinate ring $A(X)$ is not the zero ring.

($\Rightarrow$). Assume that $A(X)$ is not an integral domain, i.e. there are non-zero $f_1, f_2 \in A(X)$ such that $f_1 f_2 = 0$. Then $X_1 = V(f_1)$ and $X_2 = V(f_2)$ are closed, not equal to X (since f_1 and f_2 are non-zero), and

$$X_1 \cup X_2 = V(f_1) \cup V(f_2) = V(f_1 f_2) = V(0) = X.$$

Hence X is reducible.

($\Leftarrow$). Assume that X is reducible, with $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subset X$. By the bijection of the relative Nullstellensatz, $I(X_i) \neq \{0\}$ in $A(X)$ for $i = 1, 2$, and so we can choose non-zero $f_i \in I(X_i)$. Then $f_1 f_2$ vanishes on $X_1 \cup X_2 = X$. Hence $f_1 f_2 = 0 \in A(X)$, i.e $A(X)$ is not an integral domain. □

Motivation

Proposition 1.21 can be extended further to characterise irreducible varieties X with their ideals. Recall that $A(X) \cong K[x_1, \ldots, x_n]/I(X)$ is an integral domain if and only if $I(X)$ is a prime ideal. Hence, the bijection of the Nullstellensatz restricts to a bijection

$$\{\text{non-empty irreducible affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\text{prime ideals in } K[x_1, \ldots, x_n]\}.$$

Of course, the relative version holds too: the set of irreducible affine subvarieties of Y bijects with the set of prime ideals in $A(Y)$ for any affine variety Y.

This suggests that irreducibility is more useful in the Zariski topology. Thus we now want to decompose an affine variety into finitely many irreducible spaces. In fact, this works for a much larger class of topological spaces:

Definition 1.22 (Noetherian topological spaces)

A topological space X is called **Noetherian** if there is no infinite strictly decreasing chain

$$X_0 \supset X_1 \supset X_2 \supset \cdots$$

of closed subsets of X.

As expected, the Zariski topology falls into this class of topological space:

Lemma 1.23

Any affine variety is a Noetherian topological space.

Proof. Let X be an affine variety. By the relative Nullstellensatz, an infinite decreasing chain $X_0 \supset X_1 \supset X_2 \supset \cdots$ of affine subvarieties of X would give rise to an infinite increasing chain

$$I(X_0) \subset I(X_1) \subset I(X_2) \subset \cdots$$

of ideals in $A(X)$, which is impossible since $A(X)$ is a Noetherian ring by Hilbert's Basis Theorem. □

Intuitively, one can guess that the following is true as well:

Lemma 1.24 (Subspaces of Noetherian spaces are Noetherian)

Let A be a subset of a Noetherian topological space X, then A is also Noetherian.

Proof. Suppose not, then we have an infinite strictly decreasing chain of closed subsets of A, which by definition of the subspace topology we can write as

$$A \cap X_0 \supset A \cap X_1 \supset A \cap X_2 \supset \cdots$$

for closed subsets $X_0, X_1, X_2, \ldots$ of X. Then

$$X_0 \supseteq X_0 \cap X_1 \supseteq X_0 \cap X_1 \cap X_2 \supseteq \cdots$$

is a decreasing chain of closed subsets of X. In contradiction to our assumption, it is also strictly decreasing, since $X_0 \cap \cdots \cap X_k = X_0 \cap \cdots \cap X_{k+1}$ for some $k \in \mathbb{N}$ would imply $A \cap X_k = A \cap X_{k+1}$ by intersecting with A. □

Combining the two lemmas, we see that **any subset of an affine variety is a Noetherian topological space**. We can now prove the desired property of Noetherian spaces: that it has a decomposition into irreducible spaces.

Proposition 1.25 (Irreducible decomposition of Noetherian spaces)

Every Noetherian topological space X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of non-empty irreducible closed subsets. If one assumes that $X_i \not\subseteq X_j$ for all $i \neq j$, then $X_1, \ldots, X_r$ are unique (up to permutation). They are called the **irreducible components** of X.

Proof. For $X = \emptyset$ the statement is obvious (with $r = 0$).

Otherwise, to prove existence, suppose there is a topological space X for which the statement is false. In particular, X is reducible, hence $X = X_1 \cup X_1'$ for X_1, X_1' closed in X. Moreover, the statement of the proposition has to be false for at least one of these two subsets, say X_1. Continuing this construction, one arrives at an infinite chain $X \supset X_1 \supset X_2 \supset \cdots$ of closed subsets, contradiction.

To show uniqueness, assume that we have two decompositions

$$X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'.$$

Then for any $i \in \{1, \ldots, r\}$ we have $X_i \subseteq \bigcup_j X_j'$, so $X_i = \bigcup_j (X_i \cap X_j')$. But X_i is irreducible, so $X_i = X_i \cap X_j'$, i.e. $X_i \subseteq X_j'$ for some j. In the same way we have $X_j' \subseteq X_k$ for some k, so $X_i \subseteq X_j' \subseteq X_k$. By assumption this is only possible for $i = k$, and consequently $X_i = X_j'$.

Hence $\{X_1, \ldots, X_r\} = \{X_1', \ldots, X_s'\}$, which means that the two decompositions agree. □

In particular, it is often useful to find the irreducible components of an affine variety:

Example 1.26 (Computation of irreducible components)

Consider the affine variety $X = V(x_1 - x_2x_3, x_1x_3 - x_2^2) \subseteq \mathbb{A}^3_{\mathbb{C}}$. Note that

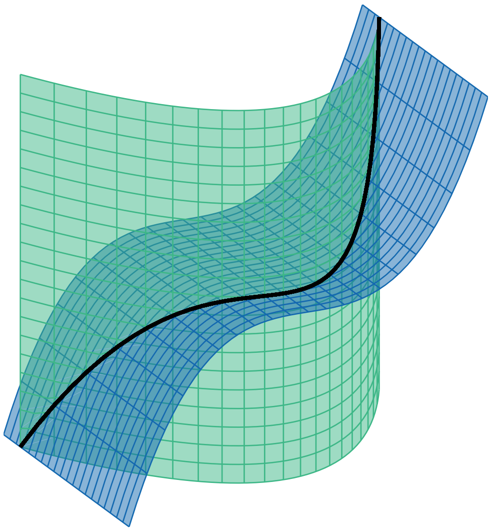
$$\begin{aligned}
(x_1, x_2, x_3) \in X &\iff x_1 = x_2x_3 \text{ and } x_1x_3 = x_2^2 \\
&\iff x_1 = x_2x_3 \text{ and } x_2x_3^2 = x_2^2 \\
&\iff x_1 = x_2x_3 \text{ and } (x_2 = 0 \text{ or } x_3^2 = x_2) \\
&\iff x_1 = x_2 = 0 \text{ or } (x_1 = x_3^3 \text{ and } x_2 = x_3^2).
\end{aligned}$$

In other words, $X = X_1 \cup X_2$ where $X_1 = V(x_1, x_2)$ and $X_2 = V(x_1 - x_3^3, x_2 - x_3^2)$. We claim that X_1, X_2 are irreducible, so they are the irreducible components of X. Indeed,

$$\begin{aligned}
A(X_1) &= \mathbb{C}[x_1, x_2, x_3]/\langle x_1, x_2 \rangle \cong \mathbb{C}[x_3] \\
A(X_2) &= \mathbb{C}[x_1, x_2, x_3]/\langle x_1 - x_3^3, x_2 - x_3^2 \rangle \cong \mathbb{C}[x_3^3, x_3^2, x_3] = \mathbb{C}[x_3]
\end{aligned}$$

both of which are integral domains. Thus by Proposition 1.21, X_1 and X_2 are irreducible, as desired.

Remark. In fact, the component X_2 is called the **twisted cubic**:



Note that it is cut out by the surfaces $V(x_1 - x_3^3)$ and $V(x_2 - x_3^2)$, as shown in the figure.

Remark. The irreducible decomposition of an affine variety $X \subseteq \mathbb{A}^n$ can also be computed from the **primary decomposition** of its ideal: If

$$I(X) = Q_1 \cap \cdots \cap Q_r$$

is a primary decomposition of $I(X)$ with Q_i primary, then by Nullstellensatz

$$X = V(I(X)) = V(Q_1) \cup \cdots \cup V(Q_r) = V(P_1) \cup \cdots \cup V(P_r)$$

where $P_i = \sqrt{Q_i}$ are prime ideals. Note that all varieties $V(P_i)$ in this union are irreducible. Keeping only the maximal varieties among them we obtain the irreducible decomposition of X. They correspond exactly to the **minimal prime ideals** in $A(X)$, so we have an additional bijection:

$$\{\text{irreducible components of } X\} \longleftrightarrow \{\text{minimal prime ideals in } A(X)\}.$$

We end the section by deploying the idea of irreducibility to show a few related results. Recall that we have already seen that open subsets tend to be very "big" in the Zariski topology. Here is the precise statements:

Lemma 1.27
Open subsets of irreducible spaces are dense.

Proof. There are multiple ways to do this; we will show that the closure $\overline{U}$ of any non-empty open subset U of an irreducible topological space X is all of X. This is easy: if $Y \subseteq X$ is any closed subset containing U then $X = Y \cup (X \setminus U)$, and since X is irreducible and $X \setminus U \neq X$ we must have $Y = X$. □

We end the section by the following result that comes up later:

Lemma 1.28
A is irreducible if and only if $\overline{A}$ is irreducible.

Proof. ($\Rightarrow$). Suppose $\overline{A}$ is reducible, i.e. $\overline{A} = A_1 \cup A_2$ for A_1, A_2 closed in $\overline{A}$. Then

$$A = A \cap \overline{A} = A \cap (A_1 \cup A_2) = (A \cap A_1) \cup (A \cap A_2).$$

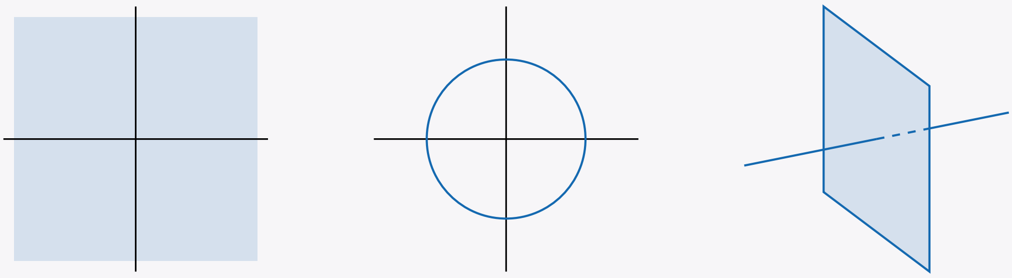
By definition of the subspace topology, $A \cap A_i$ is closed in A for $i = 1, 2$. If $A \cap A_i = A$ then $A \subseteq A_i$, so $\overline{A} \subseteq \overline{A_i} = A_i \subseteq \overline{A}$ and thus $\overline{A} = A_i$, contradiction. This shows that A is reducible. ($\Leftarrow$) is similar. □

1.5 Dimension

Developing a good theory of dimension is a challenging problem in any branch of mathematics, and algebraic geometry is no exception.

Motivation

Of course, at least in the case of complex varieties, we have a geometric idea what the dimension should be: the number of coordinates that you need to describe X locally around any point. Some intuitive examples:



$\dim \mathbb{A}^2 = 2$ $\qquad$ $\dim V(x^2+y^2-1) = 1$ $\qquad$ $\dim V(xy, xz) = 2$

The last variety has two components: $V(x)$ and $V(y,z)$. In this case, we will soon see that its dimension should be defined as the maximal dimension of the components.

The standard definition of dimension that we will give here uses only the language of topological spaces:

Definition 1.29 (Dimension and codimension)

Let X be a non-empty topological space.

(a) The **dimension** $\dim X \in \mathbb{N} \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}$ such that there is a chain

$$\emptyset \neq Y_0 \subset Y_1 \subset \cdots \subset Y_n \subseteq X$$

of irreducible closed subsets $Y_1, \dots, Y_n$ of X, where there are n strict inclusions.

(b) If $Y \subseteq X$ is a non-empty irreducible closed subset of X the **codimension** $\operatorname{codim}_X Y$ of Y in X is again the supremum over all n such that there is a chain

$$Y \subseteq Y_0 \subset Y_1 \subset \cdots \subset Y_n \subseteq X$$

of irreducible closed subsets $Y_1, \dots, Y_n$ of X containing Y.

To avoid confusion, we will always denote the dimension of a K-vector space V by $\dim_K V$.

According to the above idea, one should imagine each Y_i as having dimension i in a maximal chain as in Definition 1.29(a), so that finally $\dim X = n$. Similarly, each Y_i in Definition 1.29(b) should have dimension $i + \dim Y$ in a maximal chain, so that $n = \dim X - \dim Y$.

Example 1.30

We can verify some of our intuitions:

- The affine space $\mathbb{A}^1$ has dimension 1, since the maximal chains are exactly

$$\emptyset \neq \{a\} \subset \mathbb{A}^1$$

for any point $a \in \mathbb{A}^1$. The codimension of $\{a\}$ in $\mathbb{A}^1$ is 1.

- However, it is not entirely obvious that $\dim \mathbb{A}^n = n$, but at least it is easy to see that $\dim \mathbb{A}^n \geq n$, by considering an increasing chain of linear subvarieties.

Caution: One might be tempted to think that the Noetherian condition ensures that $\dim X$ is always finite. This is not true however: If we equip the natural numbers $X = \mathbb{N}$ with the topology in which (except $\emptyset$ and X) exactly the subsets $Y_n := \{0, \dots, n\}$ for $n \in \mathbb{N}$ are closed, then X is Noetherian, but has chains $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ of non-empty irreducible closed subsets of arbitrary length.

However, for affine varieties infinite dimensions cannot occur, since in this case the two notions reduce to two algebraic concepts:

Lemma 1.31 (Dimension and codimension of affine varieties)

Let Y be a non-empty irreducible subvariety of an affine variety X.

(a) The dimension $\dim X$ of X is equal to the **Krull dimension** of the coordinate ring $A(X)$.

(b) The codimension $\operatorname{codim}_X Y$ of Y in X is equal to the **height** of the prime ideal $I(Y)$ in $A(X)$.

In particular, dimensions and codimensions of (irreducible) affine varieties are always finite.

Proof. Both statements follow from the order-reversing property of $I(\cdot)$. The dimension and codimension must be finite since $A(X)$ is a finitely generated K-algebra. □

In fact, this correspondence allows us to transfer many results on Krull dimensions immediately to statements on dimensions of affine varieties, including $\dim \mathbb{A}^n = n$. We will list them only for irreducible varieties, since we will quickly see that the general case follows easily.

Proposition 1.32 (Properties of dimension)

Let X and Y be non-empty irreducible affine varieties.

(a) We have $\dim(X \times Y) = \dim X + \dim Y$ (note that $X \times Y$ is also an affine variety). In particular, $\dim \mathbb{A}^n = n$.

(b) If $Y \subseteq X$ then $\dim X = \dim Y + \operatorname{codim}_X Y$. In particular, $\operatorname{codim}_X \{a\} = \dim X$ for every point $a \in X$.

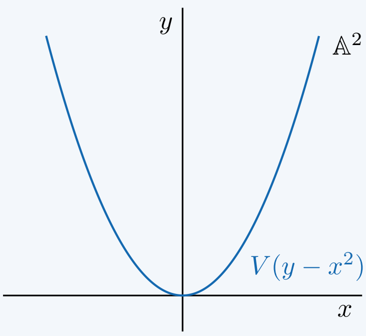
(c) If $f \in A(X)$ is non-zero then every irreducible component of $V(f)$ has codimension 1 in X (and hence dimension $\dim X - 1$ by (b)).

We omit the proof here since it is purely algebraic, and the tools used (Noether normalisation lemma and Krull's prinicpal ideal theorem) are out of the scope of this notes.

Example 1.33

Consider again the affine variety $X = V(y - x^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$. Then, as expected, we have:

- X is irreducible by Proposition 1.21 since its coordinate ring
$$A(X) = \mathbb{C}[x, y]/(y - x^2) \cong \mathbb{C}[x]$$
is an integral domain.
- X has dimension 1 by Proposition 1.32(c), since it is the zero locus of one non-zero polynomial in $\mathbb{A}^2$, and $\dim \mathbb{A}^2 = 2$.



Dimensions often behave nicely; for instance, if A is a subset of a topological space X, then $\dim A \leq \dim X$. The proof is similar to that of Lemma 1.24: If $Y_0 \subset Y_1 \subset \cdots \subset Y_n \subseteq A$ is a chain of irreducible closed subsets of A, then by definition $Y_i = A \cap X_i$ for some closed subsets X_i in X. These X_i must be irreducible since so is Y_i. Hence $X_0 \cap \cdots \cap X_n \subset X_0 \cap \cdots \cap X_{n-1} \subset \cdots \subset X_0 \subseteq X$ is a chain of irreducible closed subsets of X, i.e. $\dim X \geq n$.

Recall also that we have only showed properties of dimensions for irreducible affine varieties. The following explains why:

Lemma 1.34
Let X be a Noetherian topological space.

(a) If $X = X_1 \cup \cdots \cup X_r$ is the irreducible decomposition of X, then
$$\dim X = \max\{\dim X_1, \dots, \dim X_r\}.$$

(b) We always have $\dim X = \sup\{\operatorname{codim}_X\{a\} : a \in X\}$.

Proof. The proofs of both statements go similarly:

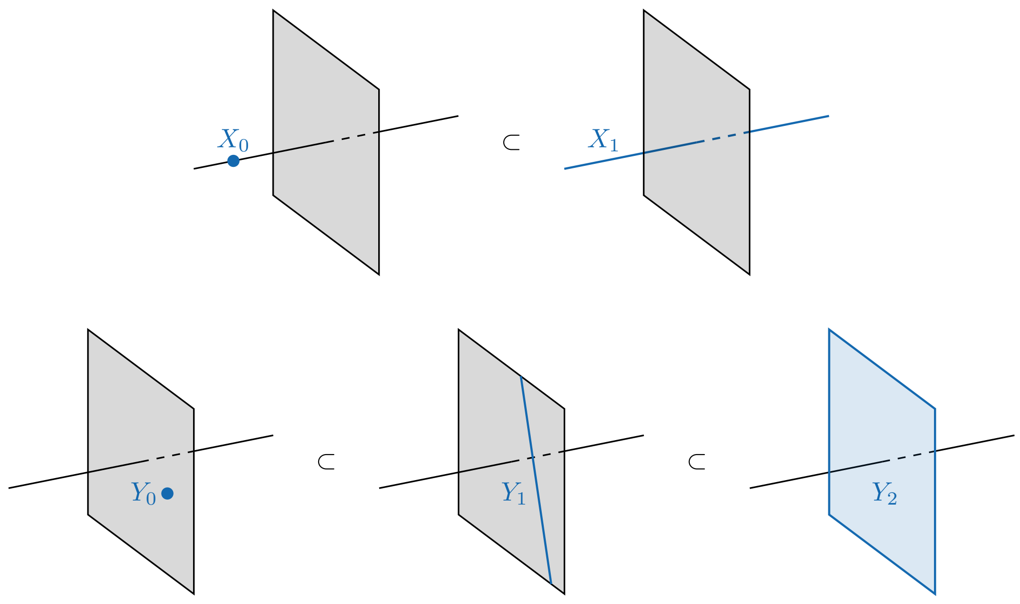
(a) $(\leq)$. Assume that $\dim X \geq n$, so there is chain $Y_0 \subset \cdots \subset Y_n$ of non-empty irreducible closed subvarieties of X. Then
$$Y_n = (Y_n \cap X_1) \cup \cdots \cup (Y_n \cap X_r)$$
is a union of closed subsets. As Y_n is irreducible, we must have $Y_n = Y_n \cap X_i$ and hence $Y_n \subset X_i$ for some i. But then $Y_0 \subset \cdots \subset Y_n$ is a chain of non-empty irreducible closed subsets in X_i, and hence $\dim X_i \geq n$.

$(\geq)$. Let $\max\{\dim X_1, \dots, \dim X_r\} \geq n$. Then there is a chain of non-empty irreducible closed subsets $Y_0 \subset \cdots \subset Y_n$ in some X_i. But this chain is also in X, and hence $\dim X \geq n$.

(b) $(\leq)$ If $\dim X \geq n$ there is a chain $Y_0 \subset \cdots \subset Y_n$ of non-empty irreducible closed subsets of X. For any $a \in Y_0$ this chain then shows that $\operatorname{codim}_X\{a\} \geq n$.

$(\geq)$. If $\operatorname{codim}_X\{a\} \geq n$ for some $a \in X$ there is a chain $\{a\} \subseteq Y_0 \subset \cdots \subset Y_n$ of non-empty irreducible closed subsets of X, which also shows that $\dim X \geq n$. $\square$

Pictorially, using the example from the motivation above, we consider $X = V(xy, xz)$:



Here $\dim X = 2$, as mentioned before, since the maximum chain of irreducible closed subvarieties occurs in the plain $V(x)$. As for (b), the codimension of the point Y_0 is 2, whereas the codimension of the point X_0 is 1, as illustrated by the chains in the picture.

As a result, the correct way to think about codimensions is:

! Keypoint

The codimension of a point $a \in X$ is the *local dimension* of X at a.

Hence Proposition 1.32(b) can also be interpreted as saying that the local dimension of an irreducible variety is the same at every point.

In practice, we will usually be concerned with affine varieties all of whose components have the same dimension. These spaces have special names we shall introduce now. Note however that these terms are not used consistently throughout the literature.

Definition 1.35 (Pure-dimensional spaces)

A Noetherian topological space X is said to be of **pure dimension** n if every irreducible component of X has dimension n.

In particular, an affine variety is called

- a **curve** if it is of pure dimension 1;
- a **surface** if it is of pure dimension 2;
- a **hypersurface** in a pure-dimensional affine variety Y if it is an affine subvariety of Y of pure dimension $\dim Y - 1$.

We have seen in Proposition 1.32(c) that the zero locus of a single polynomial in an irreducible affine variety is a hypersurface. Let us now address the opposite question: is every irreducible hypersurface of a given irreducible affine variety X the zero locus of a single polynomial?

Surprisingly, this depends on a rather subtle algebraic property of $A(X)$, based on the following result from commutative algebra:

Proposition 1.36

Let R be a Noetherian integral domain (e.g. $A(X)$ where X is an irreducible affine variety). Then R is a unique factorisation domain if and only if every prime ideal of height 1 in R is principal.

Proof. ($\Rightarrow$). Let P be a prime ideal of height 1 in R. We can then choose a non-zero element $f \in P$; since $P \neq \langle 1 \rangle$ it will also not be a unit.

As R is a unique factorisation domain, we can write

$$f = f_1 f_2 \cdots f_k$$

for some prime elements $f_1, \dots, f_k \in R$. Since P is a prime ideal we must then have $f_i \in P$ for some i. We thus obtain a chain $\{0\} \subset \langle f_i \rangle \subseteq P$ of prime ideals. But as the height of P is 1 this requires $P = \langle f_i \rangle$.

($\Leftarrow$). We first decompose any non-zero non-unit $f \in R$ as a product of irreducible elements since R is Noetherian: Otherwise f cannot be irreducible, so we must have a decomposition $f = f_1 f_1'$ into non-units, of which at least one factor, say f_1, is not a product of irreducible elements. Continuing this process, we obtain an infinite chain

$$\langle f \rangle \subset \langle f_1 \rangle \subset \langle f_2 \rangle \subset \cdots$$

in contradiction to R being Noetherian.

Now it suffices to show that every irreducible element $f \in R$ is prime. Firstly choose a minimal prime ideal P containing f. By Krull's principal ideal theorem, $\operatorname{ht} P = 1$, so by assumption P is principal, i.e. $P = \langle g \rangle$ for some prime g. But g divides f and f is irreducible, so f and g agree up to units. Hence f is prime as well. $\square$

Translating this to our case of affine varieties, if X is an irreducible hypersurface in $\mathbb{A}^n$, the $I(X) \trianglelefteq K[x_1, \ldots, x_n]$ is a prime ideal of codimension 1. As the polynomial ring is a unique factorisation domain, $I(X) = \langle f \rangle$ for an irreducible polynomial f by the above proposition.

Even if X is not irreducible, we can apply the same argument to each component of its irreducible decomposition $X = X_1 \cup \cdots \cup X_k$ to obtain $I(X_j) = \langle f_j \rangle$ for some $f_j \in K[x_1, \ldots, x_n]$ and all j. Then $I(X) = \langle f \rangle$, which is again principal. In summary:

! Keypoint

$X \subseteq \mathbb{A}^n$ is an affine hypersurface if and only if $I(X) = \langle f \rangle$.

This f is unique up to units, so we can associate its degree natually to X:

Definition 1.37 (Degree of an affine hypersurface)

Let X be an affine hypersurface in $\mathbb{A}^n$, with ideal $I(X) = \langle f \rangle$. Then the degree of f is also called the **degree** of X, denoted $\deg X$.

The general case of hypersurfaces as subvarieties in a fixed affine variety Y is much harder, since it depends on whether $A(Y)$ is a unique factorisation domain. We give one example here in which this is not the case (so there is an irreducible codimension-1 hypersurface whose ideal is not principal) to end the section:

Example 1.38

Let $R = K[x, y, z, w]/\langle xy - zw \rangle$. Then:

- R is an integral domain of dimension 3: one can show that $xy - zw$ is an irreducible element by casework. Now as $K[x, y, z, w]$ is a domain, we have in general that

  $$\operatorname{ht} I + \dim(K[x, y, z, w]/I) = \dim K[x, y, z, w] = 4.$$

  for any ideal $I \trianglelefteq K[x, y, z, w]$. Putting $I = \langle xy - zw \rangle$ (which has height 1 by Krull's principal ideal theorem), we have $\dim R = 3$.

- x, y, z, w are irreducible but not prime in R: if $x = rs$ for $r, s \in R$, then one of r, s must be degree 1 and the other one degree 0. But degree 0 polynomials are units, so x must be irreducible.

  On the other hand,

  $$R/\langle x \rangle \cong K[x, y, z, w]/\langle x, xy - zw \rangle \cong K[y, z, w]/\langle z, w \rangle$$

  which is not an integral domain. Thus x is not prime, and similarly for y, z, w.

  Remark. In particular R is not a unique factorisation domain.

- $\langle x, z \rangle$ is a prime ideal of height 1 in R that is not principal: clearly $\langle x, z \rangle$ is prime since

  $$R/\langle x, z \rangle \cong K[x, y, z, w]/\langle x, z, xy - zw \rangle \cong K[y, w].$$

  Now using the result from above again (since R is a domain) we have

  $$\operatorname{ht}\langle x, z \rangle = \dim R - \dim(R/\langle x, z \rangle) = 3 - 2 = 1,$$

  and finally $\langle x, z \rangle \neq \langle f \rangle$ for any f or else f divides x and z, which are irreducible and non-associated. So f must be an unit, but $\langle x, z \rangle \neq R$, contradiction.

Hence, the plane $V(x, z)$ is a hypersurface in the affine variety $X = V(xy - zw)$ whose ideal cannot be generated by one element in $A(X)$.

2 Sheaves & Morphisms

Having defined affine varieties, the next goal must be to say what kind of maps between them we want to consider as morphisms, i.e. the "nice" maps. This chapter answers this question.

2.1 Regular functions

The easiest case of maps is the so-called **regular functions**, i.e. the maps to the ground field $K = \mathbb{A}^1$.

Motivation

So what kind of maps do we want to consider on an affine variety X? Here is the thought process:

- Of course, any function in $A(X)$ should be considered as "nice".
- But, analogous to continuous or differentiable functions, we should not aim for a definition of functions on all of X, but also **on an arbitrary open subset** U of X.

  This allows us to consider quotients $\frac{g}{f}$ of polynomial functions $f, g \in A(X)$ with $f \neq 0$ as well, since we can exclude the zero set $V(f)$ from the domain of definition.
- But taking only rational functions is too restrictive: the problem would be that **it is not local**. In the case of continuous or differentiable function, the condition can be checked at every point. Being a quotient is however not a condition of this type – we would have to find one **global** representation as a quotient.

The way out is to consider functions that are "locally" a quotient, which gives the desired definition:

Definition 2.1 (Regular functions)

Let X be an affine variety, and let U be an open subset of X. A **regular function** on U is a map $\phi : U \to K$ with the following property: For every $a \in U$ there are polynomial functions $f_a, g_a \in A(X)$ with $f_a(x) \neq 0$ and

$$\phi(x) = \frac{g_a(x)}{f_a(x)}$$

for all x in an open subset U_a with $a \in U_a \subseteq U$. The set of all such regular functions on U will be denoted $\mathcal{O}_X(U)$; it is clearly a K-algebra.

Remark. We will usually write the condition "$\phi(x) = \frac{g_a(x)}{f_a(x)}$ for all $x \in U_a$" simply as "$\phi = \frac{g_a}{f_a}$ on U_a".

Note that f_a and g_a depends on the choice of a. Thus:

! Keypoint

ϕ is regular on U if it is locally a rational function.

This definition is misleadingly complicated, and the examples should illuminate it significantly. Firstly, most of the time we will be able to find a "global" representation, and we will not need to fuss with the a's. For example:

Example 2.2 (Regular functions)

- Any function $f \in A(X)$ is clearly regular, since we can take $f_a = 1, g_a = f$ for every a. So $A(X) \subseteq \mathcal{O}_X(U)$ for any open set U.
- Let $X = \mathbb{A}^1, U = X \setminus \{0\}$. Then $1/x \in \mathcal{O}_X(U)$ is regular on U.
- Let $X = \mathbb{A}^1, U = X \setminus \{1, 2\}$. Then

  $$\frac{1}{(x-1)(x-2)} \in \mathcal{O}_X(U)$$

  is regular on U.

The "local" clause with a's is still necessary, though:

Example 2.3 (Local $\neq$ global quotients)

Consider the affine variety

$$X = V(xy - zw) \subseteq \mathbb{A}^4$$

and the open set $U = X \setminus V(y, w) = \{(x, y, z, w) \in \mathbb{A}^4 : y \neq 0 \text{ or } w \neq 0\}$. Then

$$\phi : U \to K, (x, y, z, w) \mapsto \begin{cases} \frac{x}{w} & \text{if } w \neq 0 \\ \frac{z}{y} & \text{if } y \neq 0 \end{cases}$$

is a regular function on U:

- It is well-defined since the defining equation for X implies $x/w = z/y$ whenever $y \neq 0$ and $w \neq 0$.
- It is regular since it is obviously locally a quotient of polynomials.

But none of the two representations can be used on all of U, since we run into divide-by-zero issues. Algebraically, this just exploits the fact that

$$A(X) = K[x, y, z, w]/\langle xy - zw\rangle$$

is not a unique factorisation domain, as we have seen in Example 1.38.

As a first result, let us prove the expected statement that zero loci of regular functions are always closed in their domain of definition.

Lemma 2.4 (Zero loci of regular functions are closed)

Let U be an open subset of an affine variety X, and let $\phi \in \mathcal{O}_X(U)$ be a regular function on U. Then

$$V(\phi) := \{x \in U : \phi(x) = 0\}$$

is closed in U.

Proof. By definition any point $a \in U$ has an open neighbourhood $U_a \subseteq U$ and $f_a, g_a \in A(X)$ (with f_a nowhere zero on U_a) for which $\phi = \frac{g_a}{f_a}$ on U_a. So the set

$$U_a \setminus V(\phi) = \{x \in U_a : \phi(x) \neq 0\} = U_a \setminus V(g_a)$$

is open in X, and hence so is their union over all a, which is just $U \setminus V(\phi)$. This means that $V(\phi)$ is closed in U. □

A consequence of this lemma is the so-called identity theorem:

Corollary 2.5 (Identity theorem for regular functions)

Let $U \subseteq V$ be non-empty open subsets of an irreducible affine variety X. If $\phi_1, \phi_2 \in \mathcal{O}_X(V)$ are two regular functions on V that agree on U, then they must agree on all of V.

Proof. The locus $V(\phi_1 - \phi_2)$ contains U and is closed in V, so it contains the closure $\overline{U}$ in V. But by Lemma 1.27 $\overline{V} = X$, which is irreducible, so V is also irreducible by Lemma 1.28. But then again by Lemma 1.27, $\overline{U} = V$, so $\phi_1 = \phi_2$ on V. □

Remark. This statement is not really surprising since the open subsets in the Zariski topology are so big; but the exact same statement is also true for holomorphic functions in $\mathbb{C}^n$ which has the Euclidean topology. In this case it is an actual theorem since the open subset U can be very small.

Still this is an example of a statement that is true in literally the same way in both algebraic and complex geometry, although the topology are very different a priori.

Let us now go ahead and compute the K-algebras $\mathcal{O}_X(U)$ in some cases. A particular important result can be obtained for the following special case:

Definition 2.6 (Distinguished open subsets)

For an affine variety X and a polynomial function $f \in A(X)$ on X we call

$$D(f) := X \setminus V(f) = \{x \in X : f(x) \neq 0\}$$

the **distinguished open subset** of f in X.

From Vakil, he suggests remembering the notation $D(f)$ as "doesn't-vanish set". Similar to $V(\cdot)$, the distinguished open subsets behave nicely with respect to intersections and unions:

- For any $f, g \in A(X)$ we have $D(f) \cap D(g) = D(fg)$, since $f(x) \neq 0$ and $g(x) \neq 0$ is equivalent to $(fg)(x) \neq 0$ for all $x \in X$. In particular, finite intersections of distinguished open subsets are again distinguished open.
- Any open subset $U \subseteq X$ is a finite union of distinguished open subsets: By definition of the Zariski topology it is the complement of an affine variety $V(f_1, \ldots, f_k)$, hence

$$U = X \setminus V(f_1, \ldots, f_k) = X \setminus (V(f_1) \cap \cdots \cap V(f_k)) = D(f_1) \cup \cdots \cup D(f_k).$$

We can therefore think of the distinguished open subsets as the "smallest" open subsets of X – in topology, the correct notion for this is that they form a basis of the Zariski topology on X.

Theorem 2.7 (Regular functions on distinguished open subsets)

Let X be an affine variety, and let $f \in A(X)$. Then

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} : g \in A(X), n \in \mathbb{N} \right\}.$$

In particular,

- setting $f = 1$ we see that $\mathcal{O}_X(X) = A(X)$, i.e. the regular functions on all of X are exactly the polynomials;
- on a distinguished open subset a regular function is always globally the quotient of two polynomials.

Proof. ($\supseteq$) is obvious, since every function of the form $\frac{g}{f^n}$ for $g \in A(X)$ and $n \in \mathbb{N}$ is clearly regular on $D(f)$.

For ($\subseteq$), let $\phi : D(f) \to K$ be a regular function. By definition we obtain for every $a \in D(f)$ a local representation $\phi = \frac{g_a}{f_a}$ for some $f_a, g_a \in A(X)$ which is valid on an open neighbourhood U_a of a in $D(f)$. Now:

- After possibly shrinking U_a, we may assume that they are distinguished open subsets $D(h_a)$ for some $h_a \in A(X)$ since U_a is an union of distinguished open subsets.
- By replacing g_a and f_a by $g_a h_a$ and $f_a h_a$, we may assume both the numerator and denominator of ϕ (which we will again call g_a and f_a) vanish on $V(h_a)$.
- Finally, this means f_a vanishes on $V(h_a)$ and does not vanish on $D(h_a) \subseteq U_a$. So h_a and f_a have the same zero locus, and we can therefore assume $h_a = f_a$.

In summary we can assume that $\phi = \frac{g_a}{f_a}$ on $D(f_a)$. Now we claim that in $A(X)$ we have

$$g_a f_b = g_b f_a \quad \text{for all } a, b \in D(f) : \tag{$*$}$$

On $D(f_a) \cap D(f_b)$ these two functions agree as $\phi = \frac{g_a}{f_a} = \frac{g_b}{f_b}$ there. On $X \setminus D(f_a) \cap D(f_b) = V(f_a) \cup V(f_b)$ both sides are zero since $f_a(x) = g_a(x) = 0$ for all $x \in V(f_a)$ and $f_b(x) = g_b(x) = 0$ for all $x \in V(f_b)$ by construction.

Now all our open neighbourhoods cover $D(f)$, i.e. $D(f) = \bigcup_{a \in D(f)} D(f_a)$. Passing to the complement,

$$V(f) = \bigcap_{a \in D(f)} V(f_a) = V(\{f_a : a \in D(f)\}),$$

and thus by Nullstellensatz we have

$$f \in I(V(f)) = I(V(\{f_a : a \in D(f)\})) = \sqrt{\langle f_a : a \in D(f) \rangle}.$$

This means that $f^n = \sum_a k_a f_a$ for some $n \in \mathbb{N}$ and $k_a \in A(X)$, summing over finitely many $a \in D(f)$. Setting $g := \sum_a k_a g_a$, we then claim that $\phi = \frac{g}{f^n}$ on all of $D(f)$: for all $b \in D(f)$ we have on $D(f_b)$ that $\phi = \frac{g_b}{f_b}$ and

$$g f_b = \sum_a k_a g_a f_b \overset{(*)}{=} \sum_a k_a g_b f_a = g_b f^n.$$

These open subsets cover $D(f)$, so all local representations are equal to $\frac{g}{f^n}$. □

Note that we used Nullstellensatz again: in fact, the statement is false without the assumption of an algebraically closed ground field, by the counterexample $\frac{1}{x^2+1}$ that is defined on all of $\mathbb{R}$ but not a polynomial function.

The above statement is also deeply linked to commutative algebra. Although we considered the quotients $\frac{g}{f^n}$ as fractions of polynomial functions, we can now also interpret them as elements of a certain localisation:

Corollary 2.8 (Regular functions as localisations)

Let X be an affine variety, and let $f \in A(X)$. Then $\mathcal{O}_X(D(f))$ is isomorphic (as a K-algebra) to the localisation $A(X)_f$ of the coordinate ring $A(X)$ at the multiplicatively closed subset $\{f^n : n \in \mathbb{N}\}$.

Proof. There is an obvious K-algebra homomorphism $A(X)_f \to \mathcal{O}_X(D(f))$ defined via

$$\frac{g}{f^n} \mapsto \frac{g}{f^n}.$$

This is in fact well-defined: if $\frac{g}{f^n} = \frac{g'}{f^m}$ as formal fractions in $A(X)_f$ then $f^k(gf^m - g'f^n) = 0$ in $A(X)$ for some $k \in \mathbb{N}$, which means that on $D(f)$ we have $gf^m = g'f^n$ since $f^k \neq 0$, and thus $\frac{g}{f^n} = \frac{g'}{f^m}$ as functions on $D(f)$.

Now this homomorphism is surjective by Theorem 2.7. It is also injective: if $\frac{g}{f^n} = 0$ as a function on $D(f)$ then $g = 0$ on $D(f)$ and so $fg = 0$ on all of X, which means that

$$f(g \cdot 1 - 0 \cdot f^n) = 0 \in A(X)$$

and thus $\frac{g}{f^n} = \frac{0}{1}$ as formal fractions in $A(X)_f$. □

Example 2.9 (Regular functions on punctured plane $\mathbb{A}^2 \setminus \{0\}$)

Probably the easiest case of an open subset of an affine variety that is not a distinguished open subset is the punctured plane $U = \mathbb{A}^2 \setminus \{0\}$ in $X = \mathbb{A}^2$. We claim that

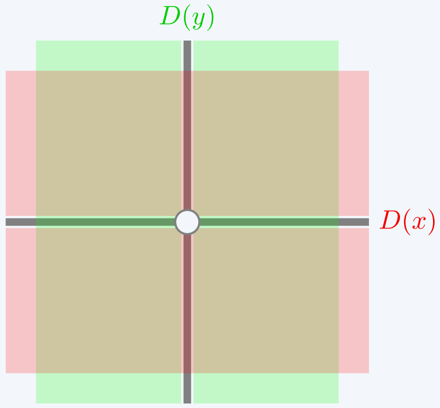
$$\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = K[x, y],$$

and thus $\mathcal{O}_X(U) = \mathcal{O}_X(X)$, i.e. every regular function on U can be extended to X.

To prove the claim, let $\phi \in \mathcal{O}_X(U)$. Note that ϕ is regular on the two distinguished open subsets

$$D(x) = (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1 \quad \text{and} \quad D(y) = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$$

since they are both subsets of U (in fact they cover U):



Hence by Theorem 2.7 we can write $\phi = \frac{f}{x^m}$ on $D(x)$ and $\phi = \frac{g}{y^n}$ on $D(y)$ for some $f, g \in K[x, y]$ and $m, n \in \mathbb{N}$. Of course we can assume that $x \nmid f$ and $y \nmid g$.

On the intersection $D(x) \cap D(y)$ both representation of ϕ are valid, so we have $fy^n = gx^m$ on $D(x) \cap D(y)$. But the locus $V(fy^n - gx^m)$ is closed, so it also contains $\overline{D(x) \cap D(y)} = \mathbb{A}^2$. In other words,

$$fy^n = gx^m \in A(\mathbb{A}^2) = K[x, y].$$

Now if $m > 0$ then x must divide fy^n, which is only possible if $x \mid f$ as $K[x, y]$ is a unique factorisation domain. This is a contradiction, so $m = 0$. But then $\phi = f$ is a polynomial, as desired.

To end the section, we note that the above example can be generalised as follows: Let Y be a non-empty irreducible subvariety of an affine variety X, and set $U = X \setminus Y$. If $A(X)$ is a unique factorisation domain and $\operatorname{codim}_X Y \geq 2$, then $\mathcal{O}_X(U) = A(X)$. The proof goes similar to above, except instead of x, y we have to choose $f_1, f_2 \in I(Y)$ using the codimension condition.

2.2 Sheaves

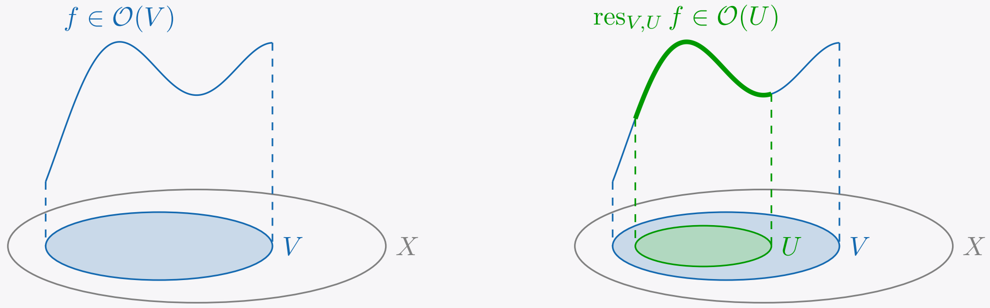
We defined regular functions on an open subset U of an affine variety as set-theroetic functions from U to the ground field K that satisfy some local property. Local constructions like this occur in many places in algebraic geometry as well as other "topological" fields, so we spend a section to formalise the idea of sheaves.

Motivation

We consider differentiable functions on the topological space $X = \mathbb{R}^n$. There are two key facts:

- On each open set $U \subset X$, we have a ring of differentiable functions, denoted by $\mathcal{O}(U)$, where
  - addition is given by $(f + g)(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})$;
  - multiplication is given by $(fg)(\mathbf{x}) := f(\mathbf{x}) \cdot g(\mathbf{x})$.
- If we have a differentiable function on an open set, we can restrict it to a smaller open set, i.e. if $U \subset V$ then we have a "restriction map"
  $$\operatorname{res}_{V,U} : \mathcal{O}(V) \to \mathcal{O}(U).$$
  The restriction maps need to "naturally" commute, i.e. if $U \subset V \subset W$, then the following commutes:
  $$\begin{tikzcd} \mathcal{O}(W) \arrow[rr, "\operatorname{res}_{W,V}"] \arrow[dr, "\operatorname{res}_{W,U}"'] & & \mathcal{O}(V) \arrow[dl, "\operatorname{res}_{V,U}"] \\ & \mathcal{O}(U) & \end{tikzcd}$$
  This reads as "restricting from big to small is equal to restricting from big to medium, then to small".

To visualise the restriction maps, it is best to "plot" the graphs, so we obtain the following figures:



To this step, we have already defined what's called a **presheaf** (the object $\mathcal{O}$).

Now to proceed to the actual sheaf we need two more technical observation:

- Let $f_1, f_2 \in \mathcal{O}(U)$, and let U_i be open sets such that $U = \bigcup_{i \in I} U_i$, i.e. $\{U_i\}$ covers U. Suppose that f_1 and f_2 agree on each of these U_i, then **they must have been the same function to begin with.**

  In other words, if $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$ (or $f_1|_{U_i} = f_2|_{U_i}$) for all i, then $f_1 = f_2$.
- Suppose we have the same U and open cover as above. Let $f_i \in \mathcal{O}(U_i)$ and assume they agree on the pairwise overlaps. Then **they can be "glued together" to form one differentiable function on** U.

  In other words, if $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ for all i, j, then there is some $f \in \mathcal{O}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$.

These two observations on differentiable functions are fundamental to the nature of $\mathbb{R}^n$, so they eventually became the axioms of a **sheaf** as we will see later on. Intuitively, the first axiom says that there is at most one way to glue together differentiable functions, and the second says there is at least one way to glue.

The major takeaway from this motivation is as follows:

! Keypoint

The entire example would have worked with continuous functions, or regular functions, or just plain functions.

So these classes of "nice" functions share common properties, and indeed these are all generalised to the properties of sheaves. We now give the actual definition:

Definition 2.10 (Presheaf)

A **presheaf** $\mathcal{F}$ (of sets) on a topological space X is the following data:

- for each open set $U \subseteq X$, we have a set $\mathcal{F}(U)$ (the elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**;
- for each inclusion $U \subseteq V$ of open sets, we have a **restriction map** $\mathrm{res}_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$.

with two requirements:

- the map $\mathrm{res}_{U,U}$ is the identity $\mathrm{id}_{\mathcal{F}(U)}$ for all U;
- if $U \subseteq V \subseteq W$, then the restriction maps in the following diagram commute:

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\mathrm{res}_{W,V}"] \arrow[dr, "\mathrm{res}_{W,U}"'] & & \mathcal{F}(V) \arrow[dl, "\mathrm{res}_{V,U}"] \\ & \mathcal{F}(U) & \end{tikzcd}$$

We sometimes also write $\operatorname{res}_{V,U} \phi$ as simply $\phi|_U$. Note that some authors also includes $\mathcal{F}(\emptyset) = \{e\}$, an arbitrary one-element set in the definition. By convention, if "over U" is omitted, then it is implicitly taken to be over X, i.e. **sections of $\mathcal{F}$** means sections of $\mathcal{F}$ over X, which are also called **global sections**.

Remark. Categorically, a presheaf is exactly the data of a contravariant functor from the category of open sets of X to Sets as morphisms in the category of open sets are precisely inclusion maps, i.e. $\mathcal{F}$ is a contravariant functor

$$\mathcal{F} : \mathsf{OpenSets}(X)^{\mathrm{op}} \to \mathsf{Sets}.$$

Definition 2.11 (Sheaf)

A **sheaf** is a presheaf that satisfies the following two axioms:

(a) Identity: If $\{U_i\}$ is an open cover of U, $\phi_1, \phi_2 \in \mathcal{F}(U)$ and $\phi_1|_{U_i} = \phi_2|_{U_i}$ for all i, then $\phi_1 = \phi_2$.

(b) Gluing: If $\{U_i\}$ is an open cover of U and $\phi_i \in \mathcal{F}(U_i)$ for all i such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ for all i, j, then there is some $\phi \in \mathcal{F}(U)$ such that $\phi|_{U_i} = \phi_i$ for all i.

Equivalently, a presheaf is a sheaf if it satisfies the "unique gluing" axiom: If $\{U_i\}$ is an open cover of U and $\phi_i \in \mathcal{F}(U_i)$ such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ for all i, j, then there is a **unique** $\phi \in \mathcal{F}(U)$ such that $\phi|_{U_i} = \phi_i$ for all i.

Remark. Although we have only constructed (pre-)sheaves of sets, in the same way we can defined (pre-)sheaves of K-algebras, abelian groups or other suitable categories, by requiring all $\mathcal{F}(U)$ are objects and all restriction maps are morphisms in the corresponding category. In the following, we will mainly be concerned with sheaves of rings, K-algebras and modules.

Example 2.12 (Example of sheaves)

Intuitively speaking, any "function-like" object form a presheaf:

- Presheaves of plain / continuous / differentiable / smooth real functions are all sheaves on $\mathbb{R}^n$, since continuity is a *local* property (meaning that we only have to look at small open neighbourhoods at once).
- Let X be an affine variety, then the rings $\mathcal{O}_X(U)$ of regular functions on open subsets $U \subseteq X$ together with the usual restriction maps of functions form a sheaf $\mathcal{O}_X$ of K-algebras on X, again because a map is regular if it is *locally* a quotient.

  We call $\mathcal{O}_X$ the **sheaf of regular functions** on X.
- However, the presheaf of constant real functions is **not** a sheaf in general, since we can pick two disjoint open sets U_1, U_2, so that the constant function 1 on U_1 and the constant function 2 on U_2 are not gluable.

The examples should ring a bell in your head about how to view a sheaf correctly:

! Keypoint

Sheaves are presheaves for which $\mathcal{F}$ is a *local* property.

In order to get used to the language of sheaves let us now consider two common constructions with them.

Definition 2.13 (Restrictions of (pre-)sheaves)

Let $\mathcal{F}$ be a presheaf on a topological space X, and let $U \subseteq X$ be an open subset. Then the **restriction** of $\mathcal{F}$ to U is defined to be the presheaf $\mathcal{F}|_U$ on U with

$$\mathcal{F}|_U(V) := \mathcal{F}(V)$$

for every open subset $V \subseteq U$ (which is also open in X), and with the restiction maps taken from $\mathcal{F}$. Note that if $\mathcal{F}$ is a sheaf then so is $\mathcal{F}|_U$.

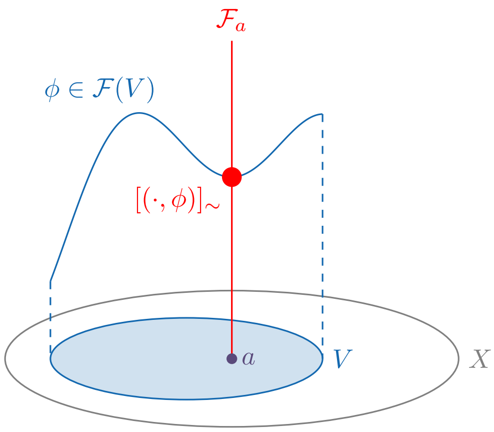
Definition 2.14 (Stalks and germs of (pre-)sheaves)
Let $\mathcal{F}$ be a presheaf on a topological space X, and let $a \in X$. Then the **stalk** of $\mathcal{F}$ at a is defined as

$$\mathcal{F}_a := \{(U, \phi) : U \subseteq X \text{ open with } a \in U, \text{ and } \phi \in \mathcal{F}(U)\}/\sim,$$

where $(U, \phi) \sim (U', \phi')$ if there is an open subset V with $a \in V \subseteq U \cap U'$ and $\phi|_V = \phi'|_V$. Note that this is an equivalence relation. The elements of $\mathcal{F}_a$ are called **germs** of $\mathcal{F}$ at a.

Remark. In the case of the sheaf $\mathcal{O}_X$ on an affine variety X, it is customary to write its stalk at a point $a \in X$ as $\mathcal{O}_{X,a}$ instead of as $(\mathcal{O}_X)_a$.

It is rarely useful to think of a germ as an ordered pair, since the set U can be arbitrarily small. Instead, one should think of a germ as a "shred" of some section near a, i.e. a germ in $\mathcal{F}_a$ stores the data of some small region near a, rather than just the value of $\phi(a)$. So accordingly we draw a large dot for the germ:



We note here that $\mathcal{F}_a$ inherits the structure of a ring (K-algebra) from $\mathcal{F}(U)$: for example, addition is done by

$$[(U_1, \phi_1)] + [(U_2, \phi_2)] := [(\phi_1|_{U_1 \cap U_2} + \phi_2|_{U_1 \cap U_2}, U_1 \cap U_2)].$$

Example 2.15 (Example of germs)

- Let $X = \mathbb{R}$ and let $\mathcal{F}$ be the sheaf on X of smooth functions. Consider a global section $f : \mathbb{R} \to \mathbb{R}$ (i.e. $f \in \mathcal{F}(X)$) and its germ at 0.
  - From the germ we can read off $f(0)$, obviously.
  - We can also find $f'(0)$, because picking an arbitrary representative of the germ gives an open set containing 0, so we can compute $\lim_{h\to 0} \frac{1}{h}[f(h) - f(0)]$. Similarly we can find $f''(0)$ and so on.
  - However, we can't read off, say $f(3)$ from the germ. For example, consider

    $$f(x) = \begin{cases} e^{-\frac{1}{x-1}} & x > 1 \\ 0 & x \leq 1. \end{cases}$$

    Note that $f(3) = e^{-\frac{1}{2}}$, but germs cannot distinguish between f and the zero function.
- However, if we let $X = \mathbb{C}$ and let $\mathcal{F}$ be the sheaf on X of holomorphic functions, then:
  - We can again read off $f(0), f'(0)$, etc.
  - But the miracle here is that just knowing the derivatives of f at zero is enough to reconstruct all of f: we can compute the Taylor series of f now, i.e. **the germs of holomorphic functions determine the entire function**.

Going back to our focus, germs of regular functions on an irreducible affine varieties also share the above property:

Lemma 2.16

Let $\phi, \psi \in \mathcal{O}_X(U)$ be two regular functions on an open subset U of an irreducible affine variety. If ϕ and ψ have the same germ on one stalk $\mathcal{O}_{X,a}$ for some $a \in U$ then $\phi = \psi$.

Proof. We have $\phi|_{U_a} = \psi|_{U_a}$ on some open neighbourhood $U_a \subseteq U$ of a, which is also open in X. But then $U_a \subseteq V(\phi - \psi)$, which is closed by Lemma 2.4, so the closure $\overline{U_a} = X$ (by Lemma 1.27) is also contained in $V(\phi - \psi)$. □

Finally, stalks of regular functions on an affine variety X can also be described algebraically in terms of localisations – which is in fact the reason why this algebraic concept is called "localisation":

Lemma 2.17 (Stalks of regular functions as localisations)

Let a be a point on an affine variety X. Then the stalk $\mathcal{O}_{X,a}$ of $\mathcal{O}_X$ at a is isomorphic (as a K-algebra) to the localisation $A(X)_{I(a)}$ of the coordinate ring $A(X)$ at the maximal ideal $I(a) \trianglelefteq A(X)$, i.e. we have

$$\mathcal{O}_{X,a} \cong \left\{ \frac{g}{f} : f, g \in A(X) \text{ with } f(a) \neq 0 \right\}.$$

In particular, $\mathcal{O}_{X,a}$ is a local ring (called the **local ring** of X at a), with unique maximal ideal

$$I_a := \{[(U, \phi)] \in \mathcal{O}_{X,a} : \phi(a) = 0\} \cong \left\{ \frac{g}{f} : f, g \in A(X) \text{ with } g(a) = 0 \text{ and } f(a) \neq 0 \right\}.$$

Proof. Consider the K-algebra homomorphism $A(X)_{I(a)} \to \mathcal{O}_{X,a}$ defined via

$$\frac{g}{f} \mapsto \left[\left(D(f), \frac{g}{f} \right) \right]$$

which makes sense since $D(f)$ is an open subset of X with $a \in D(f)$, and $\frac{g}{f} \in \mathcal{O}_X(D(f))$ by Theorem 2.7. This is well-defined: if $\frac{g}{f} = \frac{g'}{f'}$ in the localisation then $h(gf' - g'f) = 0$ for some $h \in A(X) \setminus I(a)$. Hence

$$\frac{g}{f} = \frac{g'}{f'} \quad \text{on } D(h) \cap D(f) \cap D(f') \ni a$$

and so they determine the same element in the stalk $\mathcal{O}_{X,a}$.

Now this map is surjective since by definition any regular function in a sufficiently small neighbourhood of a must be representable by a fraction $\frac{g}{f}$ with $g \in A(X)$ and $f \in A(X) \setminus I(a)$. It is also injective: suppose $\frac{g}{f}$ represents the zero element in the stalk $\mathcal{O}_{X,a}$, i.e. it is zero in an open neighbourhood of a. From which we may assume by shrinking that this open neighbourhood is a distinguished open subset $D(h)$ containing a, i.e. with $h \in A(X) \setminus I(a)$. But then $h(g \cdot 1 - 0 \cdot f)$ is zero on all of X, hence zero in $A(X)$, so $\frac{g}{f} = \frac{0}{1}$ in $A(X)_{I(a)}$. □

To end, we give an equivalent categorical definition of a stalk for category theory lovers:

Definition 2.18 (Stalks and germs, categorical)

A **stalk** of $\mathcal{F}$ at a point p is a colimit of all $\mathcal{F}(U)$ over open neighbourhoods U of p:

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U).$$

The image of $f \in \mathcal{F}(U)$ under the colimit map is then called the **germ** of f at p.

The two definitions are equivalent since the index category $\mathsf{OpenSets}(X)$ is a **filtered** set. Thus this definition actually allows us to define stalks for sheaves of sets, groups, rings, and other things for which colimits exist.

2.3 Morphisms

So far we have defined and studied regular functions on an affine variety X. They can be thought of as the morphisms from open subsets of X to the ground field $K = \mathbb{A}^1$. We now want to extend this notion of morphisms to other affine varieties than just to $\mathbb{A}^1$.

Definition 2.19 (Ringed spaces)

A **ringed space** is a topological space X together with a sheaf of rings on X. In this situation the given sheaf will always be denoted $\mathcal{O}_X$ and called the **structure sheaf** of the ringed space.

Hence, from now on, an affine variety will always be considered as a ringed space together with its sheaf of regular functions as the structure sheaf. Moreover, an open subset U of a ringed space X will always be considered as a ringed space with the structure sheaf being the restriction $\mathcal{O}_X|_U$.

Motivation

So, how might we define the "nice" maps between affine varieties? Well, we have the idea that regular functions make up the structure of an affine variety, so the obvious idea is to define a morphism

$$f : X \to Y$$

between affine varieties (or more generally ringed spaces) to be maps preserving this structure in the sense that for any regular $\phi : U \to K$ on an open subset of Y, the composition

$$\phi \circ f : f^{-1}(U) \xrightarrow{f} U \xrightarrow{\phi} K$$

is again a regular function.

Remark. A slight technical problem: the elements of $\mathcal{O}_X(U)$ or $\mathcal{O}_Y(V)$ might not be functions since the definition has to work for general ringed spaces, so composition might not make sense. To fix this, we from now on assume that sheaves of rings are actually sheaves of K-valued functions, i.e. if $\mathcal{F}$ is a sheaf of rings then $\mathcal{F}(U)$ is a subring of the ring of all functions from U to K.

With this convention we can now go ahead to define morphisms:

Definition 2.20 (Pullbacks and morphisms)

Let $f : X \to Y$ be a map of ringed spaces.

(a) For any map $\phi \in \mathcal{O}_Y(U)$ we define the **pull-back** of ϕ by f, denoted $f^*\phi$, to be the composition

$$f^*\phi = \phi \circ f : f^{-1}(U) \xrightarrow{f} U \xrightarrow{\phi} K.$$

(b) The map f is a **morphism** (of ringed spaces) if it is continuous, and if for all open subsets $U \subseteq Y$ and $\phi \in \mathcal{O}_Y(U)$ we have $f^*\phi \in \mathcal{O}_X(f^{-1}(U))$.

f is an **isomorphism** if it has a two-sided inverse which is also a morphism.

Caution: The requirement of f being continuous in necessary: or else $\mathcal{O}_X(f^{-1}(U))$ is not well-defined.

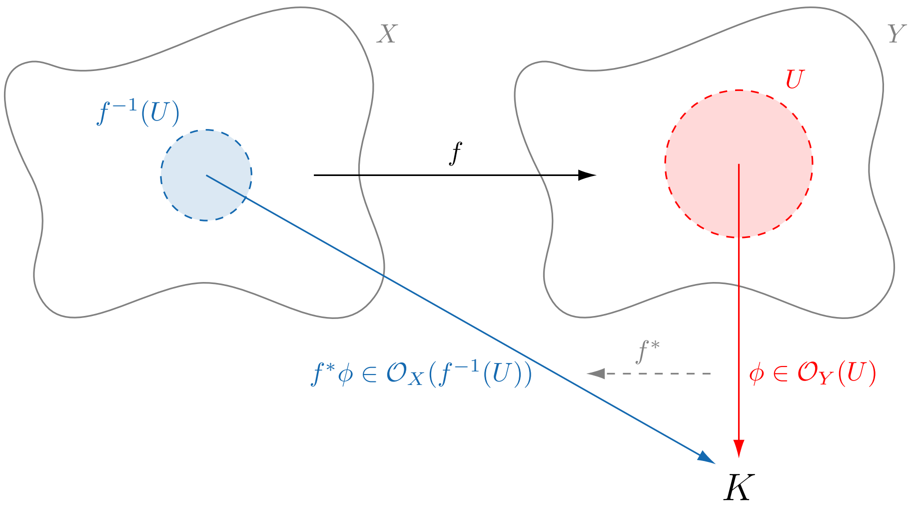
In particular, when f is a morphism, the pull-back gives us a K-algebra homomorphism

$$f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$

for *every* U; thus our morphisms package a lot of information.

Remark. Without the assumption that $\mathcal{O}_Y(U)$ contains functions, one would actually have to include suitable ring homomorphisms $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. In other words, in this case a morphism would no longer be just a set-theoretic map satisfying certain properties. This happens for schemes that we will discuss later.

Here's a picture of a morphism f, and the pull-back of $\phi : U \to K$:



Example 2.21 (Example of pull-back)

The map

$$f : X = \mathbb{A}^1 \to Y = \mathbb{A}^1, t \mapsto t^2$$

is a morphism of varieties. For example, consider the regular function $\phi(y) = \frac{1}{y-25}$ on the open set $Y \setminus \{25\} \subseteq Y$. The pre-image under f is $X \setminus \{\pm 5\}$, thus the pull-back is

$$\begin{aligned} f^*\phi : X \setminus \{\pm 5\} &\to Y \setminus \{25\} \\ x &\mapsto \frac{1}{x^2 - 25} \end{aligned}$$

which is regular on $X \setminus \{\pm 5\}$.

As some simple observations, we have:

- Compositions of morphisms are morphisms: If $f : X \to Y$ and $g : Y \to Z$ are morphisms of ringed spaces then so is $g \circ f : X \to Z$, since it is clearly continuous and for $\phi \in \mathcal{O}_Z(U)$ we have

$$(g \circ f)^*\phi = \phi \circ g \circ f = f^*(g^*\phi) \in \mathcal{O}_X(f^{-1}(g^{-1}(U))) = \mathcal{O}_X((g \circ f)^{-1}(U)).$$

- Restrictions of morphisms are morphisms: If $f : X \to Y$ is a morphism of ringed spaces and $U \subseteq X, V \subseteq Y$ are open subsets such that $f(U) \subseteq V$ then the restricted maps $f|_U : U \to V$ is again a morphism of ringed spaces.

Conversely, morphisms satisfy a "gluing" property similar to that of a sheaf:

Lemma 2.22 (Gluing property for morphisms)

Let $f : X \to Y$ be a map of ringed spaces. Assume that there is an open cover $\{U_i : i \in I\}$ of X such that all restrictions $f|_{U_i} : U_i \to Y$ are morphisms. Then f is a morphism.

Proof. By definition we have to check two things:

- f is continuous: Let $V \subseteq Y$ be an open subset. Then

$$f^{-1}(V) = \bigcup_{i \in I}(U_i \cap f^{-1}(V)) = \bigcup_{i \in I}(f|_{U_i})^{-1}(V).$$

But all restrictions $f|_{U_i}$ are continuous so $(f|_{U_i})^{-1}(V)$ are open in U_i, and hence open in X, i.e. this is a union of open sets. Thus f is continuous.

- f pulls back sections of $\mathcal{O}_Y$ to sections of $\mathcal{O}_X$: Let $V \subseteq Y$ be an open subset and $\phi \in \mathcal{O}_Y(V)$. Then
$$(f^*\phi)|_{U_i \cap f^{-1}(V)} = (f|_{U_i \cap f^{-1}(V)})^*\phi \in \mathcal{O}_X(U_i \cap f^{-1}(V))$$
since $f|_{U_i}$ (and thus also $f|_{U_i \cap f(V)}$ by restricting) is a morphism. By the gluing property for sheaves, this means that $f^*\phi \in \mathcal{O}_X(f^{-1}(V))$. □

Let us now apply our definition of morphisms to affine varieties. The following proposition can be viewed as a confirmation that our constructions above were reasonable:

Proposition 2.23 (Morphisms between affine varieties)

Let U be an open subset of an affine variety X, and let $Y \subset \mathbb{A}^n$ be another affine variety. Then the morphisms $f : U \to Y$ are exactly the maps of the form
$$f = (\phi_1, \ldots, \phi_n) : U \to Y, x \mapsto (\phi_1(x), \ldots, \phi_n(x))$$
with $\phi_i \in \mathcal{O}_X(U)$ for all $i = 1, \ldots, n$.
In particular, the morphisms from U to $\mathbb{A}^1$ are exactly the regular functions in $\mathcal{O}_X(U)$.

Proof. First assume that $f : U \to Y$ is a morphism. For $i = 1, \ldots, n$ the i-th coordinate function y_i on $Y \subseteq \mathbb{A}^n$ is clearly regular on Y, and so
$$\phi_i := f^* y_i \in \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(U)$$
by definition. But this is just the i-th component function of f, and so $f = (\phi_1, \ldots, \phi_n)$.

Conversely, let now $f = (\phi_1, \ldots, \phi_n)$ with $\phi_1, \ldots, \phi_n \in \mathcal{O}_X(U)$ and $f(U) \subseteq Y$.

- f is continuous: Let Z be closed in Y. Then $Z = V(g_1, \ldots, g_m)$ for some $g_1, \ldots, g_m \in A(Y)$, and
$$f^{-1}(Z) = \{x \in U : g_i(\phi_1(x), \ldots, \phi_n(x)) = 0 \text{ for all } i = 1, \ldots, m\}.$$
But the functions $x \mapsto g_i(\phi_1(x), \ldots, \phi_n(x))$ are regular on U, since locally plugging in quotients of polynomial functions gives again locally a quotient of polynomial functions. Hence $f^{-1}(Z)$ is closed in U by Lemma 2.4.
- f pulls back sections of $\mathcal{O}_Y$ to sections of $\mathcal{O}_X$: Let $\phi \in \mathcal{O}_Y(W)$ be a regular function on some open subset $W \subset Y$. Then
$$f^*\phi = \phi \circ f : f^{-1}(W) \to K, x \mapsto \phi(\phi_1(x), \ldots, \phi_n(x))$$
is regular again by the same argument. Thus f is a morphism. □

For affine varieties themselves (rather than their open subsets) we obtain as a consequence the following useful corollary that translates our geometric notion of morphisms entirely into the language of commutative algebra:

Corollary 2.24

For any two affine varieties X and Y, there is a bijection
$$\begin{aligned} \{\text{morphisms } X \to Y\} &\longleftrightarrow \{K\text{-algebra homomorphisms } A(Y) \to A(X)\} \\ f &\longmapsto f^*. \end{aligned}$$
In particular, isomorphisms of affine varieties correspond exactly to K-algebra isomorphisms in this way.

Proof. By definition it is clear that any morphism $f : X \to Y$ determines a K-algebra homomorphism
$$f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X),$$
i.e. $f^* : A(Y) \to A(X)$ by Theorem 2.7.

Conversely, let $g : A(Y) \to A(X)$ be a K-algebra homomorphism. Assume that $Y \subseteq \mathbb{A}^n$ and denote by $y_1, \dots, y_n$ the coordinate functions of $\mathbb{A}^n$. Then $\phi_i := g(y_i) \in A(X) = \mathcal{O}_X(X)$ for all $i = 1, \dots, n$. If we set $f = (\phi_1, \dots, \phi_n) : X \to \mathbb{A}^n$ then we obtain for any $h \in K[y_1, \dots, y_n]$

$$(f^*h)(x) = h(f(x)) = h(\phi_1, \dots, \phi_n) \overset{(*)}{=} g(h)(x) \quad \text{for all } x \in X,$$

where $(*)$ holds since both sides of the equation are K-algebra homomorphisms in h, and putting in $h = y_i$ for $i = 1, \dots, n$ (i.e. a set of generators of $K[y_1, \dots, y_n]$) we see that

$$y_i(\phi_1, \dots, \phi_n) = \phi_i(x) = g(y_i)(x).$$

This shows that $h(f(x)) = g(h)(x) = 0$ for all $h \in I(Y)$ since these polynomials are zero in $A(Y)$. Hence the image of f lies in $V(I(Y)) = Y$, i.e. we have a map $f : X \to Y$. As its coordinate functions are regular, it is indeed a morphism by Proposition 2.23, and moreover the above shows $f^* = g$ so we have the desired bijection.

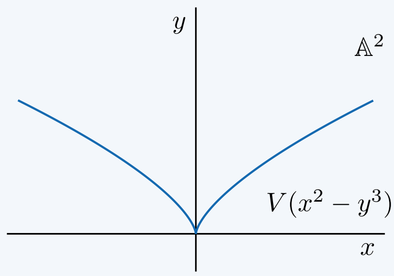
The additional statement about isomorphisms now follows immediately since $(f \circ g)^* = g^* \circ f^*$ and $(g \circ f)^* = f^* \circ g^*$ for all $f : X \to Y$ and $g : Y \to X$. □

That being said, one has to be very careful when dealing with isomorphisms of ringed spaces:

Caution: An isomorphism of ringed spaces is not necessarily a bijective morphism.

Example 2.25 (Isomorphisms $\neq$ bijective morphisms)

Let $X = V(x^2 - y^3) \subseteq \mathbb{A}^2$ be a cubic curve:



Consider the map

$$f : \mathbb{A}^1 \to X, t \mapsto (t^3, t^2)$$

which is a morphism by Proposition 2.23. Its corresponding K-algebra homomorphism f^* is

$$\begin{aligned} f^* : K[x,y]/(x^2 - y^3) &\to K[t] \\ x &\mapsto f^*x = x \circ f = t^3 \\ y &\mapsto f^*y = y \circ f = t^2. \end{aligned}$$

Now note that f is bijective with inverse map

$$(x, y) \mapsto \begin{cases} \frac{x}{y} & \text{if } x_2 \neq 0 \\ 0 & \text{if } x_2 = 0. \end{cases}$$

But f is not an isomorphism (i.e. f^{-1} is not a morphism), since otherwise by Corollary 2.24 f^* has to be an isomorphism as well – which is false since clearly

$$\deg f^*p \geq 2$$

for any $p \in K[x,y]/(x^2 - y^3)$, so for instance t is not in the image of f^*.

Another consequence of Proposition 2.23 concerns the definition of the product $X \times Y$ of two affine varieties X and Y. Recall from Example 1.18 that $X \times Y$ does not carry the product topology. The following however justifies this choice, by showing that the definition satisfies the so-called **universal property**.

> **Proposition 2.26 (Universal property of products)**
> Let X and Y be affine varieties, and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projection morphisms from the product to the two factors. Then for every affine variety Z and two morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$ there is a unique morphism $f : Z \to X \times Y$ such that $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$.

The picture is

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}
$$

and in other words, this means

> **! Keypoint**
> Giving a morphism to $X \times Y$ is the same as giving a morphism to each of the factors X and Y.

Proof. Obviously, the only way to obtain the relations $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$ is to take the map $f : Z \to X \times Y$ by $z \mapsto (f_X(z), f_Y(z))$. But this is clearly a morphism by Proposition 2.23: as f_X and f_Y must be given by regular functions in each coordinate, the same is then true for f. $\square$

Moreover, recall that taking $A(-)$ reverses the morphism arrows, i.e. we have a commutative diagram

$$
\begin{tikzcd}
R & & \\
& A(X \times Y) \arrow[ul, dashed] & A(X) \arrow[l] \arrow[ull, bend right] \\
& A(Y) \arrow[u] \arrow[uul, bend left] &
\end{tikzcd}
$$

This is a diagram of a **coproduct**. In the category of K-algebras, this correspond to the notion of **tensor products**. Hence, the coordinate ring $A(X \times Y)$ is just the tensor product $A(X) \otimes_K A(Y)$.

We now bravely proceed to a very important realisation:

> **Motivation**
> Corollary 2.24 suggests a different way to construct affine varieties: Let R be a finitely generated K-algebra, and assume that it is **reduced**, i.e. it has no nilpotent elements. We can then pick generators $a_1, \dots, a_n$ for R and obtain a surjective K-algebra homomorphism
> $$g : K[x_1, \dots, x_n] \to R, f \mapsto f(a_1, \dots, a_n).$$
> We then have $R \cong K[x_1, \dots, x_n]/ \ker g$, and furthermore $\ker g$ is radical since R is reduced. Hence $X = V(\ker g)$ is an affine variety in $\mathbb{A}^n$, with coordinate ring R.

Note that this construction of X from R depends on the choice of generators, so we get different affine varieties. However, Corollary 2.24 implies that all these affine varieties will be isomorphic since they have isomorphic coordinate rings, but they just **differ in their embeddings** in affine spaces.

This motivates us to make a redefinition of the term "affine variety" to allow for objects that are isomorphic to an affine variety but do not come with an intrinsic description as the zero locus of some polynomials:

Definition 2.27 (Redefinition of affine varieties)

From now on, an **affine variety** will be a ringed space that is isomorphic to an affine variety in the old sense.

Remark. With this new definition, the above motivation can be reformulated by saying that there is a bijection

$$\{\text{affine varieties}\}/\text{isomorphisms} \longleftrightarrow \{\text{finitely generated reduced } K\text{-algebras}\}/\text{isomorphisms}$$

that **also extends to morphisms** by Corollary 2.24.

Note also that all our concepts carry immediately to an affine variety X in this new sense:

- All topological concepts are still defined since X is a topological space.
- Regular functions are just sections of the structure sheaf $\mathcal{O}_X$.
- The coordinate ring $A(X)$ can be considered to be $\mathcal{O}_X(X)$ (by Theorem 2.7).
- Products involving X can be defined using any embedding of X in an affine space (yielding a product that is unique up to isomorphisms).

Yet this redefinition still introduces new objects – for instance, the most important example of affine varieties in this new sense that do not look like affine varieties a priori are the distinguished open subsets:

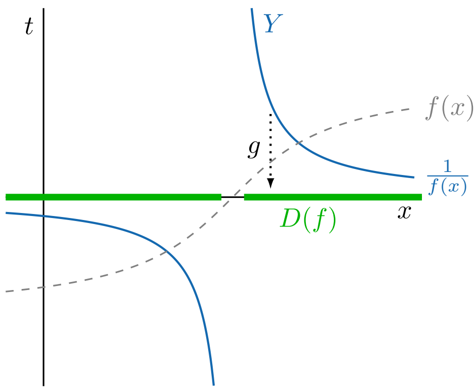
Proposition 2.28 (Distinguished open subsets are affine varieties)

Let X be an affine variety, and let $f \in A(X)$. Then the distinguished open subset $D(f)$ is an affine variety with coordinate ring $A(D(f)) \cong A(X)_f$.

Proof. Consider

$$Y := \{(x,t) \in X \times \mathbb{A}^1 : tf(x) = 1\} \subseteq X \times \mathbb{A}^1$$

which is an affine variety as it is the zero locus of the polynomial $tf(x) - 1$ in the affine variety $X \times \mathbb{A}^1$:



The affine variety Y is isomorphic to $D(f)$ by the projection morphism

$$g : Y \to D(f), (x,t) \mapsto x$$
$$\text{with inverse} \quad g^{-1} : D(f) \to Y, x \mapsto \left(x, \frac{1}{f(x)}\right)$$

so $D(f)$ is an affine variety. By Theorem 2.7 we also have $A(D(f)) \cong \mathcal{O}_X(D(f)) \cong A(X)_f$. $\square$

To end the section, we give one example which, even in the new sense, is not an affine variety:

Example 2.29 (Punctured plane is not affine)

As in Example 2.9 let $X = \mathbb{A}^2$ and consider the open subset $U = \mathbb{A}^2 \setminus \{0\}$ of X. Then even in the new sense the ringed space U is not an affine variety:

- Otherwise its coordinate ring would be $\mathcal{O}_X(U)$, and thus just the polynomial ring $K[x, y]$ by Example 2.9.
- But this is the equal to $A(X)$, and hence Corollary 2.24 would imply that U and X are isomorphic, with the isomorphism given by the identity map.

This is obviously not true, so U is not an affine variety.

However, as mentioned before, we can cover U by the two (distinguished) open subsets

$$D(x) = \{(x, y) : x \neq 0\} \quad \text{and} \quad D(y) = \{(x, y) : y \neq 0\}$$

which are affine by Proposition 2.28. This leads to the idea that we should also consider ringed spaces that can be **patched together from affine varieties**. We will do that in the next chapter.

3 Varieties

In this chapter we will finally introduce the main objects of study, the so-called *varieties*.

Motivation

If you know what a manifold is, recall that to construct them one first considers open subsets of $\mathbb{R}^n$ that are supposed to form the patches of your space, and then defines a manifold to be a topological space that looks locally like these patches.

This is completely analogous in our algebraic case: the affine varieties form the basic patches, and general varieties are then spaces that **look locally like affine varieties**.

One of the main reason for this is that in the classical topology affine varieties over $\mathbb{C}$ are never compact, unless they are finite. As compact spaces are often better-behaved, we would like to have a method to compactify an affine variety by "adding some points at infinity". This will be done when we construct projective varieties.

3.1 Prevarieties

Let us start by defining spaces that can be covered by affine varieties:

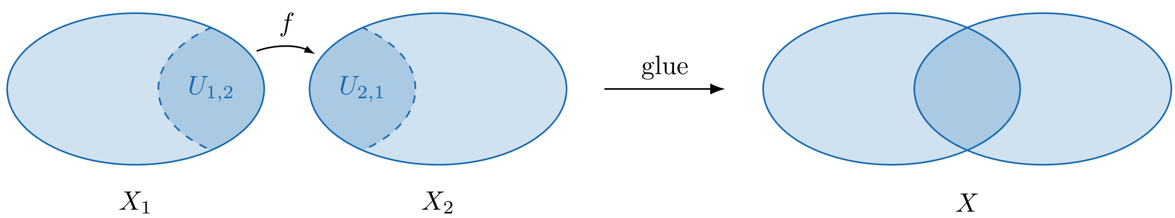
Definition 3.1 (Prevarieties)

A **prevariety** is a ringed space X that has a finite open cover by affine varieties. Morphisms of prevarieties are simply morphisms as ringed spaces, and the elements of $\mathcal{O}_X(U)$ for an open subset $U \subset X$ will be called **regular functions** on U.

Remark. Note that the open cover is not part of the data needed to specify a prevariety; it is just required that such a cover exists. Any open subset of a prevariety that is an affine variety is called an **affine open set**.

From this definition, it is clear that any affine variety is a prevariety. More generally, every open subset of an affine variety is a prevariety: they are covered by distinguished open subsets which are affine open sets.

The basic way to construct new prevarieties is to glue them together from previously known patches. Let X_1, X_2 be two prevarieties, and let $U_{1,2} \subseteq X_1, U_{2,1} \subseteq X_2$ be open subsets. Moreover, let $f : U_{1,2} \to U_{2,1}$ be an isomorphism. Then we can define a prevariety X by **gluing** X_1 and X_2 along f:



- As a set, the space X is just the disjoint union $X_1 \cup X_2$ modulo the equivalence relation given by $a \sim f(a)$ and $f(a) \sim a$ for all $a \in U_{1,2}$ (in addition to $a \sim a$ for all $a \in X_1 \cup X_2$). Note that this gives natural embeddings

$$i_1 : X_1 \to X \quad \text{and} \quad i_2 : X_2 \to X$$

  by sending a point to its equivalence class in $X_1 \cup X_2$.

- As a topological space, we call a subset $U \subseteq X$ open if $i_1^{-1}(U) \subseteq X_1$ and $i_2^{-1}(U) \subseteq X_2$ are open. In topology, this is called the **quotient topology** of i_1 and i_2.

- As a ringed space, the structure sheaf $\mathcal{O}_X$ by

$$\mathcal{O}_X(U) = \{\phi : U \to K : i_1^*\phi \in \mathcal{O}_{X_1}(i_1^{-1}(U)) \text{ and } i_2^*\phi \in \mathcal{O}_{X_2}(i_2^{-1}(U))\}$$

  for any open subset $U \subseteq X$. Intuitively, this means a function on the glued space is regular if it is regular when restricted to both patches. Clearly this defines a sheaf on X.

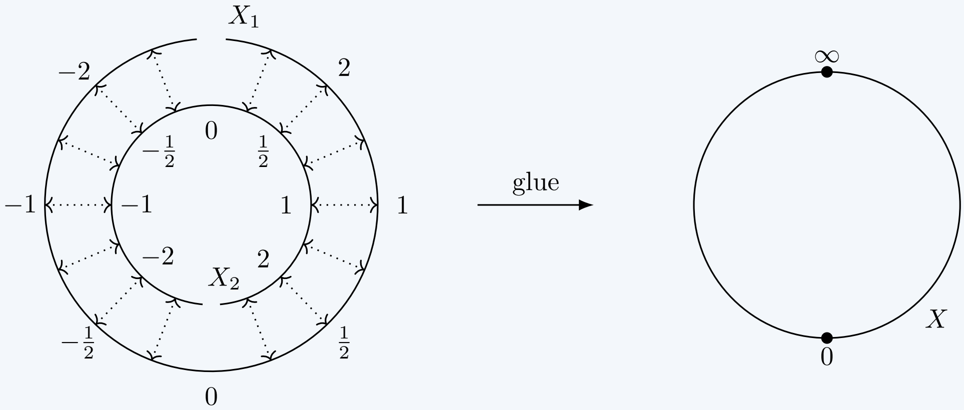
With this construction it is checked that the images of i_1 and i_2 are open subsets of X that are isomorphic to X_1 and X_2. We will often drop the inclusion maps and say X_1, X_2 are open subsets of X. Since X_1 and X_2 are covered by affine open subsets, the same is true for X, so X is a prevariety.

Example 3.2 (Gluing two copies of $\mathbb{A}^1$)

As the simplest example, let $X_1 = X_2 = \mathbb{A}^1$ and $U_{1,2} = U_{2,1} = \mathbb{A}^1 \setminus \{0\}$. We consider two different choices of the gluing isomorphism:

- Let $f : U_{1,2} \to U_{2,1}, x \mapsto \frac{1}{x}$. We have $X \setminus X_1 = X_2 \setminus U_{2,1}$ which is a single point corresponding to 0 in X_2, and so to "$\infty = \frac{1}{0}$" in the coordinate of X_1. Hence we can think of the glued space X as $\mathbb{A}^1 \cup \{\infty\}$, and thus as a "compactification" of the affine line. We denote it by $\mathbb{P}^1$.

  In the case $K = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C} \cup \{\infty\}$:

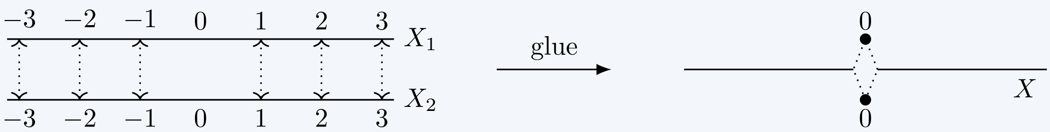


  As an example of gluing morphisms as in Lemma 2.22, the morphisms

$$X_1 \to X_2 \subset \mathbb{P}^1, x \mapsto x \quad \text{and} \quad X_2 \to X_1 \subset \mathbb{P}^1, x \mapsto x$$

  (which correspond to a reflection across the horizontal axis in the picture above) glue together to a single morphism $\mathbb{P}^1 \to \mathbb{P}^1$ that can be thought of as $x \mapsto \frac{1}{x}$, if we think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$.

- Let $f : U_{1,2} \to U_{2,1}$ be the identity map. Then the space X obtained by gluing X_1 and X_2 along f is shown in the picture below, it is "an affine line with two zero points":



  Obviously this is a somewhat weird space. Speaking in analytic terms in the case $K = \mathbb{C}$, a sequence of points tending to zero would have two possible limits in X, namely the two zero points.

  Also, as in (a) the two morphisms

$$X_1 \to X_2 \subset \mathbb{P}^1, x \mapsto x \quad \text{and} \quad X_2 \to X_1 \subset \mathbb{P}^1, x \mapsto x$$

  glue again to a morphism $g : X \to X$; this time it exchanges the two zero points and thus the set

$$\{x \in X : g(x) = x\} = \mathbb{A}^1 \setminus \{0\}$$

  is not closed in X, despite being given by an equality of continuous maps.

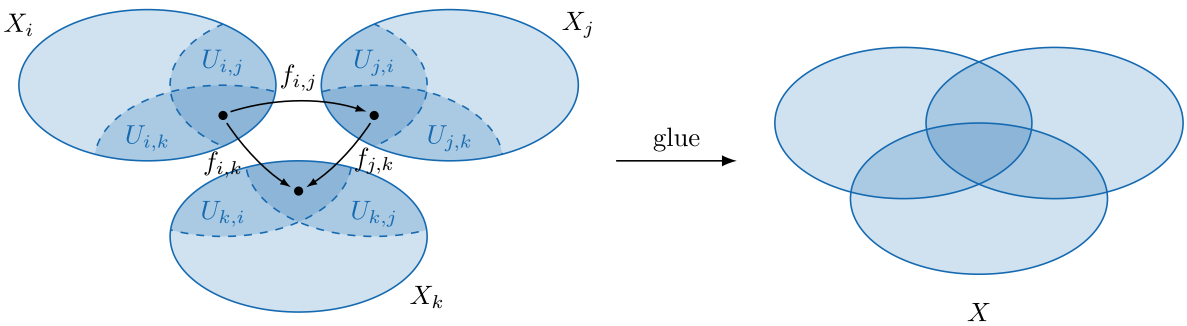
  Usually we want to exclude such spaces from the objects we consider. This will be fixed in the next section.

Let us now turn to the general construction to glue more than two spaces together. In principle this works the same way as before; we just have an additional technical compatibility condition.

For a finite index set I let X_i be a prevariety for all $i \in I$, and suppose for all $i, j \in I$ with $i \neq j$ we have open subsets $U_{i,j} \subseteq X_i$ and isomorphisms $f_{i,j} : U_{i,j} \to U_{j,i}$ such that for all distinct $i, j, k \in I$ we have

(a) $f_{j,i} = f_{i,j}^{-1}$;

(b) $U_{i,j} \cap f_{i,j}^{-1}(U_{j,k}) \subseteq U_{i,k}$ and $f_{j,k} \circ f_{i,j} = f_{i,k}$ on $U_{i,j} \cap f_{i,j}^{-1}(U_{j,k})$.



Remark. Note that the set-theoretic condition in (b) just says that the domain of definition of $f_{j,k} \circ f_{i,j}$ is included in the domain of definition of $f_{i,k}$, so that $f_{j,k} \circ f_{i,j} = f_{i,k}$ makes sense.

In analogy to gluing two prevarieties, we can then define a set X by taking the disjoint union of all X_i for all $i \in I$, modulo the equivalence relation $a \sim f_{i,j}(a)$ for all $a \in U_{i,j} \subseteq X_i$. In fact, the two conditions (a) and (b) above ensure precisely that this relation is symmetric and transitive, respectively. We can also endow a topology and a structure sheaf on X in a similar fashion.

We will devote the remaining section to study some of the basic properties of prevarieties. Of course, all topological concepts (connectedness, irreducibility and dimension etc.) carry over to the case of prevarieties.

Definition 3.3 (Open and closed subprevarieties)

Let X be a prevariety.

(a) Let $U \subseteq X$ be an open subset. Then U is again a prevariety (with structure sheaf $\mathcal{O}_U$ as in Definition 2.13): As X can be covered affine varieties, U can be covered by open subsets of affine varieties, which themselves can be covered by affine varieties.

We call U (with this structure as a prevariety) an **open subprevariety** of X.

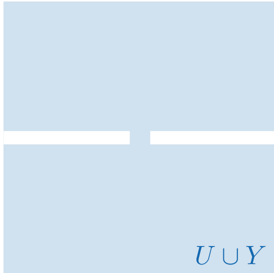
(b) For a closed subset $Y \subset X$ the situation is more complicated: We cannot simply define $\mathcal{O}_Y(U)$ as $\mathcal{O}_X(U)$ since U is in general not open in X. Instead, we define $\mathcal{O}_Y(U)$ to be the K-algebra of functions $U \to K$ that are **locally restrictions of functions on** X, or formally

$$\mathcal{O}_Y(U) := \{\phi : U \to K : \text{for all } a \in U \text{ there is an } \overbrace{\text{open neighbourhood } V \subseteq X \text{ of } a}^{\text{locally}}$$
$$\underbrace{\text{and } \psi \in \mathcal{O}_X(V) \text{ with } \phi = \psi \text{ on } U \cap V}_{\text{restrictions of functions on } X}\}.$$

By the local nature of this definition $\mathcal{O}_Y$ is a sheaf, thus making Y a ringed space (and also a prevariety) called a **closed subprevariety** of X.

Remark. Note that the extra local condition in the case of a closed subprevariety is required not only to make $\mathcal{O}_Y$ a sheaf: we need to artificially create an open neighbourhood V of X so that $\mathcal{O}_X(V)$ makes sense.

Unfortunately, for a general subset of X there is no way to make it into a prevariety in a natural way. Even worse, the notions of open and closed subprevarieties do not mix very well: for example, in $X = \mathbb{A}^2$ consider the union of the open subprevariety $U = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$ and the closed subprevariety $Y = \{0\}$:



This subset does not have a natural structure as a subprevariety of $\mathbb{A}^2$, since it does not look like an affine variety in a neighbourhood of the origin.

Example 3.4 (Morphisms on closed subprevarieties)

Since we have defined regular functions on a closed subprevariety Y of X, we might discuss morphisms from and to Y. For instance:

- The inclusion map $i : Y \to X$ is a morphism, since it is continuous, and if $\phi \in \mathcal{O}_X(U)$ then
$$i^*\phi = \phi \circ i = \phi|_Y$$
which is clearly regular, and so $i^*\phi \in \mathcal{O}_Y(i^{-1}(U)) = \mathcal{O}_Y(U \cap Y)$.
- If $f : Z \to X$ is a morphism from a prevariety Z such that $f(Z) \subseteq Y$ then we can regard f as an morphism from Z to Y, since pull-back of a regular function on Y by f is locally also a pull-back on X.

Remark. As with continuous maps, the image of an open or closed subprevariety under a morphism is not necessarily an open or closed subprevariety. For instance, consider the prevariety $X = V(yz - 1) \cup \{0\} \subseteq \mathbb{A}^3$ and the morphism
$$f : X \to \mathbb{A}^2, (x, y, z) \mapsto (x, y).$$
Then the image $f(X)$ is exactly the space $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$ which we have seen is neither open nor closed.

As a substitute, one can often consider the graph of f instead of its image, see Proposition 3.8(a).

As for the product $X \times Y$ of two prevarieties X and Y, the natural idea is to construct this space by choosing affine covers $\{U_i\}$ and $\{V_j\}$ of X and Y, and then gluing the affine product varieties $U_i \times V_j$. This is essentially correct, except we have to prove that this does not depend on the chosen affine cover. The best way out of this trouble is to use a universal property:

Proposition 3.5 (Products of prevarieties)

A product $X \times Y$ (satisfying the universal property as in Proposition 2.26) of two prevarieties X and Y exist and is unique up to unique isomorphism.

Recall that the picture is:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

and as before, this means that giving a morphism to $X \times Y$ is the same as giving a morphism to X and Y.

Proof. Note that uniqueness come directly from the universal property. We simply show existence here.

Let X and Y be covered by affine varieties $U_1, \dots, U_n$ and $V_1, \dots, V_m$ respectively. We glue any two affine products $U_i \times V_j$ and $U_{i'} \times V_{j'}$ along the identity isomorphism of the common open subset

$$(U_i \cap U_{i'}) \times (V_j \cap V_{j'}).$$

Note that these isomorphisms clearly satisfy the two needed conditions of the construction, and that the resulting space is set-theoretically $X \times Y$. Moreover, using Lemma 2.22 we can the glue the affine projections

$$U_i \times V_j \to U_i \subseteq X \quad \text{and} \quad U_i \times V_j \to V_j \subseteq Y$$

to morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$.

It remains to check the universal property for this construction. If $f_X : Z \to X$ and $f_Y : Z \to Y$ are any two morphisms from a prevariety Z, the only way to achieve $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$ is to define

$$f : Z \to X \times Y, f(z) = (f_X(z), f_Y(z)).$$

Again by Lemma 2.22, we can check that this is a morphism by restricting it to an affine open cover: first cover Z by the open subsets $f_X^{-1}(U_i) \cap f_Y^{-1}(V_j)$, and these subsets then by affine open subsets, so we may assume that every affine subset in our open cover of Z is mapped to a single (and hence affine) patch $U_i \times V_j$. But then it follows from Proposition 2.26 that f is a morphism. □

Remark. Note that there are two structures of prevarieties on the product of two closed subprevarieties $X' \subseteq X$ and $Y' \subseteq Y$: the closed subprevariety and the product prevariety structure. But these agree: the set-theoretic identity map is a morphism between these two structures in both ways.

3.2 Separatedness

Let us now impose a condition on prevarieties that finally defines varieties in general:

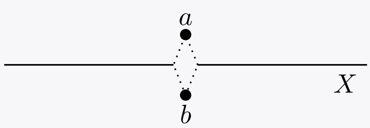
Motivation

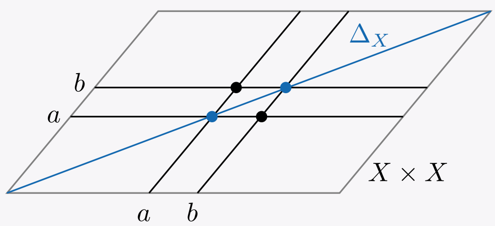
The goal is to exclude prevarieties such as the affine line with two zero points. In the theory of manifolds, this is done by requiring that the topological space satisfies the so-called **Hausdorff property**, but this is not useful in our case: any two open subsets of an irreducible space can never be disjoint.

The solution to this is inspired by a proposition in general topology stating that the Hausdorff property is equivalent to the condition that the **diagonal**

$$\Delta_X = \{(x, x) : x \in X\}$$

is closed in $X \times X$ (with the product topology). When X is the affine line with two zero points a and b:





The product $X \times X$ then contains four zero points $(a, a), (a, b), (b, a)$ and (b, b), but by definition only two of them are in Δ_X. Hence the diagonal is not closed.

Of course, this does not really help us directly in algebraic geometry since we do not use the product topology on $X \times X$; but the geometric idea to detect doubled points is still valid – and this becomes the definition.

Definition 3.6 (Varieties)

A prevariety X is called a **variety** (or **separated**) if the diagonal

$$\Delta_X := \{(x, x) : x \in X\}$$

is closed in $X \times X$.

So the affine line with two origins is not a variety. In contrast, the following lemma shows that most prevarieties that we will meet are varieties:

Lemma 3.7

(a) Affine varieties are varieties.

(b) Open and closed subprevarieties of varieties are varieties. We will therefore simply call them **open** and **closed subvarieties** respectively.

Proof. (a) If $X \subseteq \mathbb{A}^n$ then $\Delta_X = V(x_1 - y_1, \ldots, x_n - y_n) \subseteq X \times X$, where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the coordinates on the two factors, respectively. Hence Δ_X is closed.

(b) If $Y \subseteq X$ is open or closed, consider the inclusion morphism $i : Y \times Y \to X \times X$ (which exists by the universal property of Proposition 3.5). As we have $\Delta_Y = i^{-1}(\Delta_X)$ and Δ_X is closed by assumption, Δ_Y is closed as well by the continuity of i. □

Hence, from now on we will almost always assume that our spaces are separated, and thus talk about varieties instead of prevarieties. To end, we give the following additional desirable properties of varieties in addition to the ones for prevarieties:

Proposition 3.8 (Proposition of varieties)

Let $f, g : X \to Y$ be morphisms of prevarieties, and assume that Y is a variety.

(a) The **graph** $\Gamma_f := \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

(b) The set $\{x \in X : f(x) = g(x)\}$ is closed in X.

Proof. (a) By the universal property of products, there is a morphism

$$(f, \mathrm{id}) : X \times Y \to Y \times Y, (x, y) \mapsto (f(x), y).$$

As Y is a variety, we know that Δ_Y is closed, and hence so is $\Gamma_f = (f, \mathrm{id})^{-1}(\Delta_Y)$ by continuity.

(b) Similarly to (a), the given set is the inverse image of the diagonal Δ_Y under the morphism $X \to Y \times Y, x \mapsto (f(x), g(x))$, and hence is closed. □

4 Projective Varieties

4.1 Topology

4.2 Ringed Space

5 Classical constructions

5.1 Grassmannians

5.2 Blowing up

5.3 Smoothness

6 Case study: 27 Lines on a Smooth Cubic Surface